LBL-27623
UCB-PTH-89/18

New Unitary Affine-Virasoro Constructions

*

M.B. Halpern, E. Kiritsis†, N.A. Obers, M. Porrati‡,

Department of Physics, University of California
and
Theoretical Physics Group, Physics Division
Lawrence Berkeley Laboratory
1 Cyclotron Road, Berkeley, California 94720

and

Jonathan P. Yamron

Institute for Theoretical Physics, SUNY,
Stony Brook, LI, New York 11794-3800

Abstract

We report a quasi-systematic investigation of the Virasoro master equation. The space of all affine-Virasoro constructions is organized by K-conjugation into *affine-Virasoro nests*, and an estimate of the dimension of the space shows that most solutions await discovery. With consistent ansätze for the master equation, large classes of new unitary nests are constructed, including 1) quadratic deformation nests with continuous conformal weights, and 2) unitary irrational central charge nests, which may dominate unitary rational central charge on compact g.

`Published in Int. J. Mod. Phys.` **A5** `(1990) 2275.`

---

*This work was supported in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098 and in part by the National Science Foundation under grant PHY85-15857.

†email address: KIRITSIS@LBL.bitnet, 42084::KIRITSIS

‡email address: PORRATI@LBL.bitnet, 42084::PORRATI

1 Introduction

Affine Lie algebra was discovered independently in mathematics [1] and physics [2]. The first representations [2] were constructed with world-sheet fermions [2, 3] to implement the proposal of current-algebraic spin and internal symmetry on the string [2]. Examples of affine-Sugawara constructions [2, 4] and coset constructions [2, 4] were also given in the first string era, as well as the vertex operator construction of fermions and $SU(n)_1$ from compactified spatial dimensions [5, 6]. The group-theoretic generalization of these constructions [7, 8, 9] and their applications to the heterotic string [10] mark the beginning of the present era. See [11, 12, 13, 14] for further historical remarks on affine-Virasoro constructions.

The original approach of Bardakci and Halpern [2, 4] was recently resurrected in the general affine-Virasoro construction $L = L^{ab} {}^*_* J_a J_b {}^*_*$ [15, 16, 17] on the currents J_a of affine g*. The resulting Virasoro master equation for the inverse inertia tensor $L^{ab} = L^{ba}$ (and the generalization to include linear terms in J_a [15, 16]) contains the Sugawara and coset constructions, the linear conformal deformations† , and a presumably large number of new constructions. Geometric identification of the master equation as an Einstein-like system on the group manifold G [16] indicates that classification of all solutions will be a formidable program.

The master equation has so far yielded two new solutions, the generalized non-compact spin-orbit construction [2, 26, 15] with generically-irrational central charge, and a continuous unitary solution on $SU(2)_4$ [17] with $c = 1$. Witten's recent nested coset constructions [27] and the unitary non-compact coset constructions with continuous central charge‡ [28] are also solutions of the master equation.

We report here a quasi-systematic investigation of the Virasoro master equation, including organizational principles (Section 2), consistent ansätze (Section 3) and large classes of new unitary solutions (Sections 4-8). In particular:

1. The space of affine-Virasoro constructions is organized by K-conjugation into *affine-Virasoro nests*, so that all Sugawara, coset and nested coset constructions are in the simple class of *Sugawara nests* based on the Sugawara construction L_g. New constructions $L_g^{\#}$ similarly generate new affine-Virasoro nests.

2. The expected number of solutions at level x of g shows that most solutions await

*Related ideas are discussed in [18]

†Linear conformal deformations [11, 19] unify and contain a) the c-fixed conformal deformations, which generalize continuous toroidal [2, 20, 13, 21, 22] and orbifold [23, 24] compactifications, and b) the c-changing conformal deformations, which generalize the Fairlie-Feigin-Fuchs construction [25].

‡In contrast to [28], our unitary constructions with irrational central charge, described below, work entirely on the Hilbert space of compact groups.

discovery. The a priori relative abundance of central charge types is

$$irrational \gg rational \tag{1.1a}$$

$$non-unitary \gg unitary \tag{1.1b}$$

$$irrational\ unitary \gg rational\ unitary \tag{1.1c}$$

with continuous solutions occurring only sporadically. In fact, unitary irrational nests on compact g are so copious in these ansätze that their number is already approaching the number of all unitary Sugawara nests. We expect that further work will confirm dominance of unitary irrational central charge over unitary rational central charge on compact g.

With consistent ansätze for the master equation, we announce the following new unitary solutions $g_x^\#$ on the currents of g_x:

1. Unitary continuous constructions with rational central charge

$$\text{Cartan}\, g^\#; \quad SU(2)_4^\#; \quad (SU(2)_x \times SU(2)_x)^\#, \ x \neq 4 \tag{1.2}$$

which we call quadratic conformal deformations. $SU(2)_4^\#$ is the solution of [17]. These $SL(2,R)$-preserving deformations generally terminate at unitarity limits which are Sugawara and coset constructions h and g/h, and, as discussed in the Appendix, each of the constructions contains a continuous family of (1,0) states across the deformation. The physical content of these examples should be compared to particular c-fixed $SL(2,R)$-preserving linear conformal deformations [11], where an analogous (1,0) phenomenon is familiar.

2. Unitary discrete constructions with generically-irrational central charge

$$(SU(2)_x)_\#^{q\geq 3}; \quad \text{simply-laced } g^\# \tag{1.3}$$

which may be the dominant unitary solution type on compact g. Simply-laced $g^\#$ is rational for $g = SU(2)$, so $SU(3)$ and $(SU(2))^3$ are the smallest manifolds on which we have found unitary irrational central charge. The chiral constructions (1.3), and their unitary irrational affine-Virasoro nests, should be considered for promotion to unitary irrational conformal field theories.

All the new solutions, including the spin-orbit constructions [2, 26, 15], involve no more than a single square root, although every case begins with a large system of coupled quadratic equations. This indicates that the master equation is a special algebraic system, with a deep structure we do not fully understand. A few phenomenological remarks in this direction are included in Section 9.

2 The Virasoro Master Equation

The general affine-Virasoro construction is [15, 17]

$$L = L^{ab} {}^*_* J_a J_b {}^*_*, \quad [L^{(m)}, L^{(n)}] = (m-n)L^{(m+n)} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \tag{2.1}$$

with symmetric normal ordering $T_{ab} = {}^*_* J_a J_b {}^*_* = T_{ba}$ on the currents of affine g [1, 2]

$$[J_a^{(m)}, J_b^{(n)}] = if_{ab}{}^c J_c^{(m+n)} + mG_{ab}\delta_{m+n,0}. \tag{2.2}$$

Analysis of the system (2.1)-(2.2) results in the Virasoro master equation and central charge

$$L^{ab} = 2L^{ac}G_{cd}L^{db} - L^{cd}L^{ef}f_{ce}{}^a f_{df}{}^b - L^{cd}f_{ce}{}^f f_{df}{}^{(a}L^{b)e},$$
$$c = 2G_{ab}L^{ab} \tag{2.3}$$

for the inverse inertia tensor $L^{ab} = L^{ba}$.The construction is completely general since g is not necessarily semi-simple or compact. In particular, to obtain levels $x_I = 2k_I/\psi_I^2$ of g_I in $g = \oplus_I g_I$ with dual Coxeter numbers $\tilde{h}_I = Q_I/\psi_I^2$, take

$$G_{ab} = \oplus_I k_I \eta^I_{ab}, \quad f_{ac}{}^d f_{bd}{}^c = -\oplus_I Q_I \eta^I_{ab} \tag{2.4}$$

where η^I_{ab} is a Killing metric of g_I. The geometric form of the master equation and central charge [16]

$$\hat{R}_{ij} + g_{ij} = g_{ij}^{(g)}, \quad c = \dim g - 4R \tag{2.5}$$

is an Einstein-like system on the group manifold G, where the inertia tensor L_{ab} defines the left invariant metric $g_{ij} = e_i^a L_{ab} e_j^b$: $\hat{R}_{ij}$ is the Ricci tensor R_{ij} plus (torsion)^2 terms, R is the curvature scalar, and $g_{ij}^{(g)}$ is the left-right invariant Sugawara metric.

We remark on some general properties of the master equation which will be useful in the analysis below:

1. $SL(2,R)$-invariance. The vacuum $|0\rangle$ of affine g satisfies

$$J_a^{m\geq 0}|0\rangle = T_{ab}^{m\geq -1}|0\rangle = W_{abc}^{m\geq -2}|0\rangle = 0 \tag{2.6}$$

   where W_{abc} is the three-current operator of [15]. It follows that all solutions of the master equation are $SL(2,R)$-invariant with $L^{m\geq -1}|0\rangle = 0$. This is particularly interesting in the comparison of quadratic with linear [11] conformal deformations, since the c-fixed linear conformal deformations must be at least linear-loaded to preserve the $SL(2,R)$-invariant vacuum.

2. The affine-Sugawara construction [2, 4, 8] L_g is

$$L_g^{ab} = \oplus \frac{\eta_I^{ab}}{2k_I + Q_I}, \quad c_g = \sum_I \frac{x_I \dim g_I}{x_I + \tilde{h}_I} \tag{2.7}$$

   for arbitrary levels of any g, and similarly for L_h when $h \subset g$.

3. K-conjugation invariance [2, 4, 15]. If L^{ab} and L_g^{ab} are solutions, then so is the K-conjugate partner $\tilde{L}$ of L,

$$\tilde{L}^{ab} = L_g^{ab} - L^{ab}, \quad \tilde{c} = c_g - c, \tag{2.8}$$

while the corresponding K-conjugate operator pairs $L^{(m)}$ and $\tilde{L}^{(n)}$ commute. The coset constructions [2, 4, 9] $\tilde{L} = L_{g/h}$ are obtained for $h \subset g$ with $L = L_h$.

4. Affine-Virasoro nests. If $g_m \supset g_{m-1} \supset ... \supset g_1 \supset g$ and $L_g^{\#}$ is any construction with central charge $c_g^{\#}$ on g, then repeated embedding by K-conjugation produces the *affine-Virasoro nest* $\{L_{(n)}[L_g^{\#}]\}$ on g_m,

$$L_{(n)}[L_g^{\#}] = L_{g_n} - L_{(n-1)}[L_g^{\#}] = \sum_{i=1}^{n}(-1)^{n+i}L_{g_i} + (-1)^n L_g^{\#},$$

$$c_{(n)}[L_g^{\#}] = \sum_{i=1}^{n}(-1)^{n+i}c_{g_i} + (-1)^n c_g^{\#} \ , \ n = 0, 1, .., m \tag{2.9}$$

associated to $L_{(0)}[L_g^{\#}] = L_g^{\#}$. An element $L_{(j)}$ of the nest commutes with its nearest neighbors $L_{(j\pm1)}$ at the operator level. We will also use the coset notation $g_m/g_{m-1}/../g_1/g^{\#}$ for the general nest since the embedding is independent of $L_g^{\#}$. The simple class of *Sugawara nests* $g_m/../g$ with $L_g^{\#} = L_g$ contains all affine-Sugawara, coset and nested coset [27] constructions, but there are many distinct nests based on new $L_g^{\#}$.

5. Vertical-horizontal structure. A two-dimensional structure emerges for affine-Virasoro space, with subgroup nesting as the vertical direction (g above h when $g \supset h$). The horizontal direction is the set of all constructions at fixed g, which contains the subset of *irreducible* constructions

$$\{L_g^{\#}\} \ : \ (L_g^{\#})_{(0)} = L_g \ , \ (L_g^{\#})_{(1)} \ , \ (L_g^{\#})_{(2)} \ , \ .... \tag{2.10}$$

which are not obtainable by nesting from below. The irreducible constructions nest upward into a cascade of solutions at large g. It is important to emphasize that even one new $(L_g^{\#})_{(1)}$ over all levels of arbitrary g generates a set of affine-Virasoro nests whose dimension equals that of all the Sugawara nests. The list (1.3) of irreducible unitary irrational constructions shows that we are already approaching this point on compact g, and more unitary $L_g^{\#}$ are expected with further work.

6. Counting. The dimension of the horizontal space is estimated as follows. The master equation on g is a set of $\dim g(\dim g + 1)/2$ quadratic equations on an equal number of components $L^{ab} = L^{ba}$ of the inverse inertia tensor. Moding out by $\dim g$ inner automorphisms of g and subtracting the trivial pair $(L_g, 0)$, we find that

$$N_K(g) = 2^{e(g)} - 1 \ , \ e(g) = \frac{1}{2}\dim g(\dim g - 1) - 1 \tag{2.11}$$

non-trivial, possibly complex, K-conjugate pairs are expected for arbitrary level of any g. The result (2.11) shows that the cascade at large g is exponential, and evaluation on small g, e.g. $N_K = 2^{27} - 1$ for $g = SU(3)$, demonstrates that a very large number of new irreducible constructions $L^{\#}_g$ await discovery. The number of distinct pairs can be less than N_K due to degeneracy and/or physical identity of constructions. The a priori relative abundance of solution types is given in (1.1).

7. Unitarity. Each K-conjugate pair is either unitary or non-unitary with possible unitary subspaces [§]. Unitary constructions are recognized by $L^{(m)\dagger} = L^{(-m)}$ when the currents $J_a^{(m)\dagger} = J_a^{(-m)}$ act in a positive-definite unitary representation of g [9], which is satisfied for real L^{ab} on compact g.

8. Spectrum. Partial spectral results are obtained for general $L^{ab} \,{}^*_* J_a J_b \,{}^*_*$ via the highest weight construction given for g, h and g/h in [12]: if $\{R^I_{(r)}\}$ are the known quantum irreducible representations of affine g with matrix representation $\{T_a\}$,

$$[J_a^{(m)}, R^I_{(r)}] = R^J_{(r+m)} (T_a)_J{}^I , \tag{2.12}$$

then the states formed with the highest non-vanishing modes $r(0)$ of R,

$$|\Delta_M\rangle \equiv R^I_{r(0)}|0\rangle\chi_{IM}, \quad L^{ab}(T_a T_b)_I{}^J \chi_{JM} = \Delta_M \chi_{IM} \tag{2.13}$$

are Virasoro highest weights of $L^{ab} \,{}^*_* J_a J_b \,{}^*_*$. The conformal weights Δ_M are the eigenvalues of the effective Hamiltonian $H = L^{ab} T_a T_b$. There is an easy explicit version of the unitarity argument [9] in this case. When L^{ab} is real on compact g, we have hermitian H, real Δ_M and unitary χ, so the states $\{|\Delta_K\rangle\}$ in (2.13) are unitary transformations of the affine highest weight states $\{R^I_{r(0)}|0\rangle\}$ of g. Positivity of the space $\{|\Delta_M\rangle\}$, and hence $\Delta_M \geq 0$, follows from positivity of the unitary affine highest weight states.

3 Consistent Ansätze for the Master Equation

The numerical equality of restrictions and unknowns in the master equation, and hence the solvability of the system, reflects closure under OPE of the operator subset $\{L^{ab} T_{ab}, \ \forall L^{ab}\}$. Similarly, a *consistent ansatz* $\{L^{ab}(ansatz)\} \subset \{L^{ab}\}$ for the master equation is one that maintains numerical equality of equations and unknowns, indicating closure of the operator subset $\{L^{ab}(ansatz) T_{ab}\}$.

[§] As examples, non-compact coset constructions and the non-compact spin-orbit constructions ($\varepsilon = -1$ in [15]) are quasi-unitary (non-unitarity controlled by signs of the Killing metric), and this class often admits unitary subspaces [2, 4, 26, 28]

We discuss consistent ansätze for level $x_g = 2k/\psi_g^2$ of g simple and compact in the Cartan-Weyl basis, for which

$$G_{ab} = k\,\eta_{ab}\ ,\ \ \eta_{AB} = \delta_{AB}\ ,\ \ \eta_{\alpha\beta} = \delta_{\alpha+\beta,0} \tag{3.1a}$$

$$f_{A\,\alpha}{}^{\alpha} = f_{\alpha\,-\alpha}{}^{A} = -i\alpha^A\ ,\ \ f_{\alpha\,\beta}{}^{\gamma} = -iN_\gamma(\alpha,\beta) \tag{3.1b}$$

$$N_\gamma(\alpha,\beta) = N_{-\beta}(-\gamma,\alpha) = -N_{-\gamma}(-\alpha,-\beta) \tag{3.1c}$$

$$\sum_\alpha \alpha^A\alpha^B = Q_g\delta^{AB}\ ,\ \ \sum_{\beta+\gamma=\alpha} N_\alpha^2(\beta,\gamma) = Q_g - 2\alpha^2 \tag{3.1d}$$

with $A,B = 1,\cdots,rank\,g$ and α,β,γ are roots of g. The general affine-Virasoro construction takes the form

$$L = \sum_{AB} L^{AB}T_{AB} + \sum_{\alpha\beta} L^{\alpha\beta}T_{\alpha\beta} + 2\sum_{\alpha A} L^{A\alpha}T_{A\alpha} \tag{3.2a}$$

$$c = 2k(\sum_A L^{AA} + \sum_\alpha L^{\alpha,-\alpha}) \tag{3.2b}$$

and we require for unitarity that

$$L^{AB} = L^{BA} = real\ ,\ \ L^{A\alpha} = L^{\alpha A} = (L^{A,-\alpha})^*\ ,\ \ L^{\alpha\beta} = L^{\beta\alpha} = (L^{-\alpha,-\beta})^* \tag{3.3}$$

so $L^{\alpha,-\alpha}$, $\forall\alpha$ is real. Among the $dim\,g$ inner automorphisms, the unitary transformation $\exp(i\sum_A \phi^A J_A^{(0)})$ further allows the choice of $rank\,g$ phases in $\{L^{\alpha\beta}, L^{A\alpha}\}$, and we explicitly choose $L^{\alpha\alpha}$ = real, $\forall\alpha$ for $SU(2)^q \times U(1)^p$ subgroups below.

The simplest consistent ansatz is entirely on Cartan g with only $L^{AB} \neq 0$, so that the master equation simplifies to

$$L^{AB} = 2k\sum_C L^{AC}L^{CB} \tag{3.4}$$

with no root operators included. There are at least two other ansätze intermediate between (3.4) and the full master equation.

The ansatz on which we focus below involves only the components $\{L^{AB}, L^{\rho,\pm\rho}\}$ in

$$L = \sum_{AB} L^{AB}T_{AB} + \sum_{\rho>0}(2L^{\rho,-\rho}T_{\rho,-\rho} + L^{\rho\rho}T_{\rho\rho} + (L^{\rho\rho})^*T_{-\rho,-\rho}) \tag{3.5a}$$

$$c = 2k(\sum_A L^{AA} + 2\sum_{\rho>0} L^{\rho,-\rho}) \tag{3.5b}$$

for some subset of roots $\{\rho\}$ of g. The set $\{\rho\}$ always includes $-\alpha$ for unitarity when $\alpha \in \{\rho\}$. The necessary condition that no root-quartet of $\{\rho\}$ satisfy

$$\alpha + \beta = \gamma \ \text{ and }\ \alpha - \beta = \delta\ ,\ \ \alpha,\beta,\gamma,\delta \in \{\rho\} \tag{3.6}$$

is obtained on substitution of the ansatz into the master equation. This condition is satisfied by the roots of simply-laced g and the master equation takes the form

$$L^{AB} = 2k\sum_C L^{AC}L^{CB} + \sum_\rho (|L^{\rho\rho}|^2 - (L^{\rho,-\rho})^2)\rho^A\rho^B + \sum_{\rho C} L^{\rho,-\rho}\rho^C L^{C(A}\rho^{B)} \tag{3.7a}$$

$$L^{\rho\rho}\left[1 - 4kL^{\rho,-\rho} - 4\chi_\rho - 2\sum_{\alpha+\beta=\rho} L^{\beta,-\beta}N_\rho^2(\alpha,\beta)\right] = \sum_{\alpha+\beta=\rho} L^{\alpha\alpha}L^{\beta\beta}N_\rho^2(\alpha,\beta) \tag{3.7b}$$

$$L^{\rho,-\rho} = 2(k-\rho^2)|L^{\rho\rho}|^2 + 2(k+\rho^2)(L^{\rho,-\rho})^2 + \sum_{\alpha+\beta=\rho}(2L^{\rho,-\rho} - L^{\beta,-\beta})L^{\alpha,-\alpha}N_\rho^2(\alpha,\beta) \tag{3.7c}$$

$$\chi_\rho \equiv \sum_{AB} L^{AB}\rho^A\rho^B, \tag{3.7d}$$

after some algebra in this case.

Another consistent ansatz keeps L^{AB}, $L^{\alpha\beta} \neq 0$ for root subsets $\{\rho\}$ which satisfy $\alpha \pm \beta \neq \gamma$ and hence $\alpha \cdot \beta = 0$, $\alpha, \beta \in \{\rho\}$. The master equation in this case

$$L^{AB} = 2k\sum_C L^{AC}L^{CB} + \sum_{\alpha\beta}|L^{\alpha\beta}|^2\alpha^A\beta^B + \sum_{\alpha C} L^{\alpha,-\alpha}\alpha^C L^{C(A}\alpha^{B)} \tag{3.8a}$$

$$L^{\alpha\beta}[1 - \sum_{AB} L^{AB}(\alpha+\beta)^A(\alpha+\beta)^B] = \sum_\rho (2k - (\alpha-\beta)\cdot\rho)L^{\alpha\rho}L^{-\rho,\beta} \tag{3.8b}$$

is a more general description of any $(SU(2))^q \times U(1)^p$ subgroup of g.

The intersection of the ansätze (3.7) and (3.8) is the simplest starting point beyond the Cartan ansatz (3.4). We define a restricted $SU(2)^q \times U(1)^p$ subsystem by keeping only L^{AB}, $L^{\rho,\pm\rho} \neq 0$ for $\{\rho\}$ any subset of the roots of $SU(2)^q \times U(1)^p$, so that $\alpha \pm \beta \neq \gamma$, $\alpha \cdot \beta = 0$, $\alpha, \beta \in \{\rho\}$. Then (3.7) may be written for $k \neq \rho^2$ as¶

$$L^{AB} = 2k\sum_C L^{AC}L^{CB} + \sum_{\rho>0}\frac{1-4\chi_\rho}{2k}\left(\frac{2\rho^A\rho^B}{k-\rho^2}\chi_\rho + \sum_C \rho^C L^{C(A}\rho^{B)}\right) \tag{3.9a}$$

$$L^{\rho,-\rho} = \frac{1}{4k}(1-4\chi_\rho)\ ,\ (L^{\rho,\rho})^2 = \frac{1}{16k^2}(1-4\chi_\rho)(1 + \frac{k+\rho^2}{k-\rho^2}4\chi_\rho) \tag{3.9b}$$

where we have solved (3.7b)-(3.7c) to obtain $L^{\rho,\pm\rho} \neq 0$ in terms of L^{AB}. Only the simple system (3.9a) on Cartan g need be solved in this ansatz, and exceptional cases with $k = \rho^2$ are examined separately with (3.7).

The same set of equations (3.9) describes a collection of submanifolds as follows: Any solution of (3.9a) for fixed values of

$$Q \equiv order(L^{AB}) = rank(SU(2)^q \times U(1)^p) = q + p \tag{3.10a}$$

¶The derivation of (3.9) from (3.7) chooses $L^{\alpha,-\alpha} = 0$ (rather than the Sugawara value $L_g^{\alpha,-\alpha} = (k+\alpha^2)^{-1}/2$) when $L^{\alpha\alpha} = 0$, so (3.9) is K-conjugation invariant only for $R \equiv dim\{\rho > 0\} = q$. The conjugate partner $\tilde{L} = L(SU(2)^q \times U(1)^p) - L$ of any solution L of (3.9) solves (3.7) on $SU(2)^q \times U(1)^p$ for all $R \leq q$.

$$R \equiv dim\{\rho > 0\} \leq q \leq Q \tag{3.10b}$$

applies uniformly over the (Q, R) space of submanifolds

$$(Q, R) : \quad SU(2)^{Q-n} \times U(1)^n \ , \ n = 0, 1, \cdots, Q - R \tag{3.11}$$

with $dim(Q, R) = Q - R + 1$. Only the simplest case, $SU(2)^q$ with $Q = R = q$ (all roots in play) is studied explicitly below, although a further remark is included at the end of Section 7.

4 Cartan $g^{\#}$

The general solution of the master equation (3.4) on Cartan g is

$$L^{AB} = \frac{1}{2k} \sum_C \Omega^{AC} \Omega^{BC} \theta_C, \quad c = \sum_A \theta_A \tag{4.1}$$

with $\Omega \in SO(rank\, g)$ in the adjoint and $\theta_A = 0$ or 1. This solution, Cartan $g^{\#}$, and its K-conjugate partner $g/\text{Cartan}\, g^{\#}$ with central charge $c_g - \sum_A \theta_A$, are our first examples of quadratic conformal deformations: Cartan $g^{\#}$ contains

$$N = c(rank\, g - c), \quad c = 1, 2, \cdots, rank\, g \tag{4.2}$$

continuous parameters for arbitrary level of simple g.

Continuous spectrum is verified for simple highest weight states of Cartan $g^{\#}$. E.g., the results

$$E_\alpha^{(-1)}|0\rangle, \quad \Delta_\alpha = x_g^{-1} \sum_A (\sum_B \alpha^B \psi_g^{-1} \Omega^{BA})^2 \theta_A \tag{4.3a}$$

$$T_{\alpha\beta}^{(-2)}|0\rangle, \quad \Delta_{\alpha\beta} = x_g^{-1} \sum_A (\sum_B (\alpha + \beta)^B \psi_g^{-1} \Omega^{BA})^2 \theta_A \ \ , \quad \alpha + \beta \neq \gamma, 0 \tag{4.3b}$$

are obtained for fixed roots α, β of g with the commutation relations of [15]. We also computed the Cartan $SU(3)_x^{\#}$ splittings of the quantum irreducible representations $3, \bar{3}$ of $SU(3)_x$, obtaining the conformal weights

$$\Delta(\theta_8 = 1) = \frac{1}{12x}((\cos\phi + \sqrt{3}\sin\phi)^2 \, , \, (\cos\phi - \sqrt{3}\sin\phi)^2 \, , \, 4\cos^2\phi) \tag{4.4}$$

from (2.13) with the Gell-Mann basis $T_a = \sqrt{\alpha^2}\lambda_a/2$ and ϕ the angle of Ω.

The structure of Cartan $g^{\#}$ will recur below in the following connections:

1. The full master equation (2.3) for simple compact g degenerates at high level into an analogous system

$$L^{ab} \simeq 2k \sum_c L^{ac} L^{cb} \simeq \frac{1}{2k} \sum_c \Omega^{ac} \Omega^{bc} \theta_c, \quad \Omega \in SO(dim\, g) \tag{4.5a}$$

$$\lim_{k\to\infty} c = 0, 1, \cdots, dim\, g \tag{4.5b}$$

when the high level behavior of the components is $L^{ab} = O(k^{-1})$. All Sugawara nests and all the new constructions below (except $SU(2)^{\#}_4$, $(SU(2)_2 \times SU(2)_2)^{\#}$) fall in this class [‖].

2. The vertex-operator construction [5, 6, 7] implies that

$$T_{\alpha,-\alpha} = \alpha^{-2}\alpha^A\alpha^B T_{AB} \quad ; \quad T_{\alpha\beta} = 0, \;\; \alpha\cdot\beta > 0 \tag{4.6}$$

for level one of simply-laced g. These identities underlie the equivalence $L(g_1) = L(Cartan\, g_1)$ [5, 29, 13] and imply here that all new constructions on the operator subset $\{T_{AB},\, T_{\alpha,\pm\alpha}\}$ will degenerate to Cartan $g^{\#}$ for level one of simply-laced g. This remark applies to all the new solutions below.

5 $SU(2)^{\#}$

The single root subansatz

$$L^{AB} = \lambda\alpha^A\alpha^B \;, \quad \alpha > 0 \tag{5.1}$$

in (3.9a) is only a basis choice for SU(2). The resulting single equation for the parameter λ

$$\lambda(k-2\alpha^2)(1-2\lambda\alpha^2(k+\alpha^2)) = 0 \;\; , \;\; k \neq \alpha^2 \tag{5.2}$$

shows a continuous solution at $k = 2\alpha^2$, in agreement with [17], and there are no other new solutions in the system. The full solution $L^{\#}_{\eta}(\lambda)$ obtained from (3.5b),(3.9b) and (5.2) is

$$L^{\alpha,-\alpha} = \frac{1}{8\alpha^2}(1-4\lambda\alpha^4) \;,\; (L^{\alpha\alpha})^2 = \frac{1}{64\alpha^4}(1-4\lambda\alpha^4)(1+12\lambda\alpha^4) \tag{5.3a}$$

$$-\frac{1}{12} \le \lambda\alpha^4 \le \frac{1}{4} \;\; , \;\; k = 2\alpha^2 \;\; , \;\; c = 1 \tag{5.3b}$$

with $\eta = \pm 1$ the sign of $L^{\alpha\alpha}$. The unitary range in (5.3b) is decided so that $L^{\alpha\alpha}$ is real, and the solution is non-unitary outside this range. We call the unitary solution $SU(2)^{\#}_4$ since it has intrinsic level $x = 2k/\alpha^2 = 4$ when measured by its own root.

$SU(2)^{\#}_4$ is closed under K-conjugation on the SU(2) submanifold

$$\tilde{L}^{\#}_{\eta}(\lambda) = L(SU(2)\,,\, k = 2\alpha^2) - L^{\#}_{\eta}(\lambda) = L^{\#}_{-\eta}(\frac{1}{6\alpha^4} - \lambda) \tag{5.4}$$

‖The distinct asymptotic master equation $L^{ac}G_{cd}L^{db} \simeq 0$ governs the high-level behavior $L^{ab} = O(k^y)$, $y > -1$. It follows that this class contains only non-compact (or non-unitary) constructions, such as the spin-orbit construction [15] with $y = -1/2$. We have not analyzed the asymptotic central charges of this class beyond the spin-orbit construction, for which c approaches integers and half-integers between 0 and $dim\, g$.

which shows that K-conjugation in this case is reflection about the center, $\lambda\alpha^4 = 1/12$, of the continuous range plus a sign change of $L^{\alpha\alpha}$. The endpoints of the deformation are known constructions, $SU(2)_4/U(1)$ at $\lambda\alpha^4 = -1/12$ and $U(1)$ at $\lambda\alpha^4 = 1/4$. There is another $U(1)'$, $SU(2)_4/U(1)'$ pair at the quarter-points $\lambda\alpha^4 = 0, 1/6$, and a countable set of points in the interior at which the coefficients are rational.

Generically continuous spectrum of the deformation is verified for the highest weight states corresponding to the quantum irreducible representations $j = 1/2$, 1, 3/2 and 2 of $SU(2)_4$. The eigenvalue problem (2.13) in this case is the completely asymmetric SU(2) top for these spins and, e.g., the conformal weights

$$\Delta(j=1) = \frac{1}{8}\left(2(1-4\lambda\alpha^4)\ ,\ 1+4\lambda\alpha^4 \pm \eta\sqrt{(1-4\lambda\alpha^4)(1+12\lambda\alpha^4)}\right) \tag{5.5}$$

are obtained for the $L^{\#}$ splitting of spin one. The weights corresponding to $j = 3/2$, 2 are also continuous and totally split , while $\Delta(j = 1/2) = 1/16$ is independent of the deformation. This spectral data suggests that $SU(2)^{\#}_4$ is a chiral version of the line of Z_2 orbifold models at $c = 1$, where the two primary fields with fixed dimension 1/16 are the twist fields of the orbifold line.

The construction is regular-embedded along with $g/SU(2)^{\#}_4$, $c = c_g - 1$ at levels

$$x_g = x\frac{\alpha^2}{\psi_g^2} = \begin{cases} 2, 4 & \text{for } SO(2n+1), Sp(n), F_4 \\ 4 & \text{other } g \end{cases} \tag{5.6}$$

across all simple g. Other affine-Virasoro nests of $SU(2)^{\#}_4$ include irregular embeddings ** and semi-simple embedding such as $SU(2)_m \times SU(2)_n/SU(2)^{\#}_{m+n=4}$.

6 $(SU(2) \times SU(2))^{\#}$

The two-root subansatz in (3.9a)

$$L^{AB} = \lambda_\alpha \alpha^A \alpha^B + \lambda_{\alpha\beta}\alpha^{(A}\beta^{B)} + \lambda_\beta \beta^A \beta^B, \quad \alpha, \beta > 0 \tag{6.1}$$

with $\alpha \pm \beta \neq \gamma$, $\alpha \cdot \beta = 0$ is a basis choice for $SU(2) \times SU(2)$. It gives

$$\lambda_\alpha(k - 2\alpha^2)(1 - 2\lambda_\alpha\alpha^2(k + \alpha^2)) = 2k(k - \alpha^2)\lambda^2_{\alpha\beta}\beta^2 \tag{6.2a}$$

$$\lambda_\beta(k - 2\beta^2)(1 - 2\lambda_\beta\beta^2(k + \beta^2)) = 2k(k - \beta^2)\lambda^2_{\alpha\beta}\alpha^2 \tag{6.2b}$$

$$\lambda_{\alpha\beta}\left(2k - \alpha^2 - \beta^2 + 4[\lambda_\alpha\alpha^2(\alpha^4 - k^2) + \lambda_\beta\beta^2(\beta^4 - k^2)]\right) = 0 \tag{6.2c}$$

for $k \neq \alpha^2$ or β^2 on substitution in (3.9a). We factor out $(SU(2)^{\#}_4)^2$ by requiring $\lambda_{\alpha\beta} \neq 0$, which linearizes (6.2c).

**The particular case $SU(3)_1/SU(2)^{\#}_4$ is included in Cartan $SU(3)^{\#}_1$.

When $\alpha^2 = \beta^2$, this system has a continuous solution for all $k \neq \alpha^2,\ 2\alpha^2$. With λ_α the deformation parameter, we obtain the eight solutions $L_{\{\eta\}}(\lambda_\alpha)$

$$\lambda_\beta = \frac{1}{2\alpha^2(k+\alpha^2)} - \lambda_\alpha \ , \ \lambda_{\alpha\beta}^2 = \frac{\lambda_\alpha(k-2\alpha^2)}{2k\alpha^2(k-\alpha^2)}(1 - 2\lambda_\alpha\alpha^2(k+\alpha^2)) \tag{6.3a}$$

$$L^{\alpha,-\alpha} = \frac{1}{4k}(1 - 4\lambda_\alpha\alpha^4) \quad , \quad L^{\beta,-\beta} = \frac{1}{4k}\left(\frac{k-\alpha^2}{k+\alpha^2} + 4\lambda_\alpha\alpha^4\right) \tag{6.3b}$$

$$(L^{\alpha,\alpha})^2 = (L^{\beta,\beta})^2 = \frac{1}{16k^2}(1 - 4\lambda_\alpha\alpha^4)\left(1 + 4\lambda_\alpha\alpha^4\frac{k+\alpha^2}{k-\alpha^2}\right) , \tag{6.3c}$$

from (6.2) and (3.9b), with $\{\eta\} \equiv (\eta_\alpha, \eta_\beta, \eta_{\alpha\beta})$ the signs of $L^{\alpha\alpha}$, $L^{\beta\beta}$ and $\lambda_{\alpha\beta}$. The following unitary ranges

$$\begin{aligned} k = \frac{1}{2}\alpha^2 \quad &: \quad 0 \leq \lambda_\alpha\alpha^4 \leq \frac{1}{12} \ , \ \frac{1}{4} \leq \lambda_\alpha\alpha^4 \leq \frac{1}{3} \\ k = \frac{3}{2}\alpha^2 \quad &: \quad -\frac{1}{20} \leq \lambda_\alpha\alpha^4 \leq 0 \ , \ \frac{1}{5} \leq \lambda_\alpha\alpha^4 \leq \frac{1}{4} \\ k = \frac{x}{2}\alpha^2 \ , \ x \geq 5 \quad &: \quad 0 \leq \lambda_\alpha\alpha^4 \leq \frac{1}{x+2} \end{aligned} \tag{6.4}$$

are then determined from the squared relations in (6.3a,c), with $x = 2k/\alpha^2$ the intrinsic level. The result for the central charge

$$c = \frac{3x}{x+2} \quad , \quad x \neq 2, 4 \tag{6.5}$$

is uniform across the series, which we call $(SU(2)_x \times SU(2)_x)^\#$, $x \neq 2, 4$.

The construction is closed under K-conjugation on the $SU(2) \times SU(2)$ submanifold

$$\begin{aligned} \tilde{L}^\#_\eta(\lambda_\alpha) &= L(SU(2)_x \times SU(2)_x) - L^\#_\eta(\lambda_\alpha) \\ &= L^\#_{-\eta}(\frac{1}{2\alpha^2(k+\alpha^2)} - \lambda_\alpha), \end{aligned} \tag{6.6}$$

which maps between disconnected ranges in (6.4). The endpoints of the deformation are coset constructions on the currents of $SU(2)_x \times SU(2)_x$ of the type $SU(2)_x \times U(1)/U(1)'$, each endpoint involving different $SU(2)'$s and $U(1)'$s in this case. With (2.13), we verified continuous behaviour of the $L^\#$ conformal weights corresponding to $(0, \frac{1}{2})$, $(\frac{1}{2}, 0)$ and $(\frac{1}{2}, \frac{1}{2})$ of $SU(2)_x \times SU(2)_x$. The construction $(SU(2)_x \times SU(2)_x)^\#$, $x \neq 2, 4$ has the central charge of $SU(2)_x$, so its physical content should be compared to the one-parameter torus deformations of $SU(2)_x$.

The simplest affine-Virasoro nests of $(SU(2)_x \times SU(2)_x)^\#$ are $L^\#$ and $L_g - L^\#$ at levels

$$x_g \neq \begin{cases} 2, 4 & \text{of } SO(4), SO(5), Sp(2) \text{ and rank } g \geq 3 \text{ (long roots)} \\ 1, 2 & \text{of } Sp(n \geq 4) \text{ (short roots)} \end{cases} \tag{6.7}$$

across simple compact g.

In this case, there are also new deformations at the exceptional point $k=\alpha^2$ $(x=2)$ which complete the series

$$(SU(2)_x \times SU(2)_x)^{\#} \;\;,\;\; x \neq 4 \tag{6.8}$$

as announced in the Introduction. Choosing $\lambda_{\alpha\beta}$ as the deformation parameter, we find sixteen unitary solutions $L^{\#}_{\eta_\rho}(\lambda_{\alpha\beta},\theta_\rho) = (SU(2)_2 \times SU(2)_2)^{\#}$,

$$(L^{\rho,\rho})^2 = \frac{1}{16\alpha^4}(1-16\lambda_{\alpha\rho}^2\alpha^8) \;,\; L^{\rho,-\rho} = \frac{\theta_\rho}{4\alpha^2} \;,\; \lambda_\rho = \frac{1-\theta_\rho}{4\alpha^4} \tag{6.9a}$$

$$-\frac{1}{4} \leq \lambda_{\alpha\beta}\alpha^4 \leq \frac{1}{4}, \quad c = 1+\frac{1}{2}(\theta_\alpha + \theta_\beta) \tag{6.9b}$$

with $\rho = \{\alpha,\beta\}$, η_ρ the sign of $L^{\rho\rho}$ and θ_α, $\theta_\beta = 0$ or 1. The relation

$$L^{\#}_{\eta_\rho}(\lambda_{\alpha\beta},\theta_\rho) + L^{\#}_{-\eta_\rho}(-\lambda_{\alpha\beta},1-\theta_\rho) = L(SU(2)_2 \times SU(2)_2) \tag{6.10}$$

shows closure of the level two construction under K-conjugation. The endpoints of the deformation are identified as various $U(1)'$s, $SU(2)\times U(1)/U(1)'$ and $SU(2)\times SU(2)/U(1)$.

With (2.13), we have computed the $L^{\#}$ conformal weights

$$\Delta(1/2,0) = \frac{1+\theta_\alpha}{16}, \quad \Delta(0,1/2) = \frac{1+\theta_\beta}{16} \tag{6.11}$$

corresponding to these spins of $SU(2)_2^\alpha \times SU(2)_2^\beta$, and verified that $\Delta(1/2,1/2)$ is continuous. The physical content of the constructions with $c=1,2$ should be compared with toroidal and orbifold models, while the construction with $c=3/2$ should be compared with the line of theories Z_2 orbifold $\times$ Ising fermion, where the fixed dimensions (6.11) correspond to the spin fields of the fermion, the twist fields of the orbifold, and their products.

7 $(SU(2))^q_{\#}$

The q-root subansatz in (3.9a)

$$L^{AB} = \sum_{i,j=1}^{q} \frac{\lambda_{ij}}{\sqrt{\rho_i^2\rho_j^2}}\rho_i^A\rho_j^B \;\;,\;\; \lambda_{ij}=\lambda_{ji} \;,\; \rho_i\cdot\rho_j = \delta_{ij}\rho_i^2 \;\;,\;\; \rho_i > 0 \tag{7.1}$$

is an appropriate basis for $(SU(2))^q$, but we discuss only the case $\rho_i^2 = \alpha^2$ for simplicity. The system (3.9a) on Cartan g becomes

$$\lambda_{ii}(k-2\alpha^2)(1-2\lambda_{ii}(k+\alpha^2)) = 2k(k-\alpha^2)\sum_{j\neq i}\lambda_{ij}^2 \tag{7.2a}$$

$$\lambda_{ij}(k-\alpha^2)[1-2(k+\alpha^2)(\lambda_{ii}+\lambda_{jj})] = 2k^2 \sum_{m\neq i,j} \lambda_{im}\lambda_{mj} \ , \ i \neq j \tag{7.2b}$$

for $k \neq \alpha^2$ with $\chi_i = \alpha^2 \lambda_{ii}$.

At this point, we study only the further subansatz with the highest possible symmetry among the roots,

$$q \geq 2: \quad \lambda_{ii} = \lambda_d \ , \ \forall\, i \ ; \ \lambda_{ij} = (-1)^{\theta(i)+\theta(j)}\lambda_o \ , \ i \neq j \ , \ \lambda_o \neq 0 \ , \tag{7.3}$$

where $\theta(i) = 0$ or 1, $i = 1,..,q$. The system (7.2) degenerates to the pair of quadratic equations

$$\begin{aligned} \lambda_d(k-2\alpha^2)(1-2\lambda_d(k+\alpha^2)) &= 2k(k-\alpha^2)(q-1)\lambda_o^2 \\ \lambda_o(k-\alpha^2)(1-4\lambda_d(k+\alpha^2)) &= 2k^2(q-2)\lambda_o^2 \end{aligned} \tag{7.4}$$

one of which is linear for $\lambda_o \neq 0$.

We obtain the family of discrete solutions

$$L^{AB} = \alpha^{-2}\left(\lambda_d \sum_{i=1}^{q} \rho_i^A \rho_i^B + \lambda_o \sum_{i\neq j}^{q} (-1)^{\theta(i)+\theta(j)} \rho_i^A \rho_j^B\right) \tag{7.5a}$$

$$L^{\rho,-\rho} = \frac{1}{4k}(1-4\lambda_d\alpha^2) \ , \ (L^{\rho\rho})^2 = \frac{1}{16k^2}(1-4\lambda_d\alpha^2)(1+4\lambda_d\alpha^2\frac{k+\alpha^2}{k-\alpha^2}) \tag{7.5b}$$

$$\lambda_d = \frac{1}{4(k+\alpha^2)}(1+\eta F) \ , \ \lambda_o = \eta F \frac{\alpha^2-k}{2k^2(q-2)} \tag{7.5c}$$

$$c = \frac{q}{2(x+2)}[3x+\eta(x-4)F] \ , \ F = \left[\frac{(q-2)^2x^3(x-4)}{q^2x^3(x-4)+64(q-1)(x-1)}\right]^{\frac{1}{2}} \tag{7.5d}$$

where $\eta = \pm 1$ and $x = 2k/\alpha^2$ is the intrinsic level of each SU(2). The signs of $\{L^{\rho\rho}\}$ may be chosen independently, and the set, which is closed under K-conjugation, contains 2^{2q} solutions. The central charge in (7.5d) is *generically irrational* across the family, although it asymptotes to correctly-bounded integers $q+1, 2q-1$ at high level, in accord with (4.5b).

The solution is also generically unitary, with $(SU(2)_x)^3$, $x = 1,3$ the only non-unitary points. Rational unitary points are as follows: $q = 2$ is a point in $(SU(2)_x \times SU(2)_x)^\#$; level four is a point in $(SU(2)_4^\#)^q$; $(SU(2)_1)^{q\geq 4}$ with $c = 1$, $q-1$ and $(SU(2)_3)^4$ with $c = 3$, $21/5$ are identifiable as points in Cartan $g^\#$ and $g/\text{Cartan}\, \text{g}^\#$ for these groups; finally, the exceptional case $x = 2$ contains known h and g/h.

This leaves the unitary generically-irrational family

$$(SU(2)_{x\geq 5})_\#^{q\geq 3}, \ \ (SU(2)_3)_\#^{q\geq 5}, \ \ c > 3 \tag{7.6}$$

which we call $(SU(2)_x)_\#^{q\geq 3}$. The central charge

$$c((SU(2)_5)_\#^3) = \frac{45}{14}\left(1-\frac{1}{3}\sqrt{\frac{5}{1637}}\right) \simeq 3.1551 \tag{7.7}$$

is the lowest irrational value in the construction.

Study of subansätze with lower symmetry than (7.3) may produce further new constructions on $(SU(2))^q$ and will also determine whether any part of the construction (7.5) is continuous. Continuous solutions are unlikely for irrational central charge since the previous deformations included points of h or g/h.

We close this section with a few remarks on $(SU(2))^q \times U(1)^p$. The appropriate basis in (3.9a) is

$$L^{AB} = \sum_{i,j}^{Q} \lambda_{ij} \hat{\Gamma}_i^A \hat{\Gamma}_j^B \;\;,\quad \hat{\Gamma}_i \cdot \hat{\Gamma}_j = \delta_{ij} \tag{7.8}$$

with $\hat{\Gamma}_i = \alpha_i/\sqrt{\alpha_i^2}$, $i = 1, 2, \cdots, q$ the positive roots of $SU(2)^q$ and $\hat{\Gamma}_i$, $i = q+1, \cdots, q+p = Q$ the basis of $U(1)^p$. We are also free to choose the number $R = dim\{\rho > 0\}$ of root pairs for which $L^{\rho,\pm\rho} \neq 0$ in (3.9a). It is clear from the basis (7.8) that a single (Q, R) set of equations (3.9) applies over the (Q, R) space of submanifolds (3.11).

The simplest example beyond our previous work is the space $(Q = 2, R = 1)$ which includes both $SU(2) \times U(1)$ and $SU(2) \times SU(2)$ with one root pair $L^{\alpha,\pm\alpha} \neq 0$. The system for this case has continuous solutions on $SU(2)_1 \times U(1)_1$ and $SU(2)_1 \times SU(2)_1$, which, being level one, are equivalent to constructions on Cartan $g^{\#}$.

8 Simply-Laced $g^{\#}$

For simply-laced g we have

$$dim\, g = (\tilde{h} + 1) rank\, g \;,\;\; \sum_{\alpha+\beta=\gamma} 1 \;= 2(\tilde{h} - 2) \;,\;\; N_\gamma^2(\alpha, \beta) = \frac{\rho^2}{2} \tag{8.1}$$

so that all terms in the consistent ansatz (3.7) are known. We explore simply-laced g only in the maximally-symmetric subansatz

$$L^{AB} = \lambda \rho^{-4} \sum_\rho \rho^A \rho^B = \lambda \rho^{-2} \tilde{h} \delta^{AB} \tag{8.2a}$$

$$L^{\rho,\pm\rho} = \rho^{-2} L_\pm \; real \;,\; \forall\, \rho \tag{8.2b}$$

for which $L = \rho^{-2}(\lambda \tilde{h} \sum_A T_{AA} + L_+ \sum_\rho T_{\rho\rho} + L_- \sum_\rho T_{\rho,-\rho})$ is the corresponding operator construction. The coupled system

$$\lambda(1 - x\tilde{h}\lambda - 2\tilde{h}L_-) = L_+^2 - L_-^2 \tag{8.3a}$$

$$L_+(1 - 2(x + \tilde{h} - 2)L_- - 4\tilde{h}\lambda - (\tilde{h} - 2)L_+) = 0 \tag{8.3b}$$

$$L_- = (x - 2)L_+^2 + (x + \tilde{h})L_-^2 \tag{8.3c}$$

$$c = x\tilde{h}\, rank\, g\, (\lambda + L_-) \tag{8.3d}$$

is obtained on substitution into (3.7). Linearization of (8.3b) for $L_+ \neq 0$ leads to a quartic equation whose four solutions are as follows.

The first K-conjugate pair of solutions

$$\lambda = 0 \ , \ L_- = \frac{1}{\tilde{h} + 2(x-1)} \ , \ L_+ = -\frac{1}{\tilde{h} + 2(x-1)} \tag{8.4a}$$

$$c = \frac{x\tilde{h}\, rank\, g}{\tilde{h} + 2(x-1)} \tag{8.4b}$$

$$\lambda = \frac{1}{\tilde{h}(\tilde{h}+x)} \ , \ L_- = \frac{x-2}{(\tilde{h}+x)(\tilde{h}+2(x-1))} \ , \ L_+ = \frac{1}{\tilde{h}+2(x-1)} \tag{8.4c}$$

$$c = \frac{x(x-1)(\tilde{h}+2) rank\, g}{(\tilde{h}+x)(\tilde{h}+2(x-1))} \tag{8.4d}$$

has rational central charges. The level one solution in (8.4c)-(8.4d), with $c = 0$ for all g, reduces to $L^{ab} = 0$ on application of the level one identities (4.6).

These constructions are unfamiliar but not new. To see this, note that the operators $\{i(E_\alpha - E_{-\alpha}) \, , \forall\, \alpha > 0\}$ in g (not necessarily simply-laced) generate a subgroup $h \subset g$ such that g/h is a symmetric space with $dim\, h = dim \Phi_+(g)$. This is the symmetric space with maximal $dim\, g/h$ at fixed g. The first solution (8.4a) is expressed in terms of these operators as

$$L_h = -\frac{\rho^{-2}}{\tilde{h}+2(x-1)} \sum_{\rho>0} {}^*_*(E_\rho - E_{-\rho})^2\, {}^*_* \tag{8.5}$$

and it is not difficult to check that this is the correctly-embedded Sugawara construction for the subgroups h in the complete list of symmetric spaces

$$\frac{SU(n)_x}{SO(n)_{2x}}\,(n \geq 4) \, , \ \frac{SU(3)_x}{SU(2)_{4x}} \ , \ \frac{SU(2)_x}{U(1)_x} \tag{8.6a}$$

$$\frac{SO(2n)_x}{SO(n)_x \times SO(n)_x} \ , \ \frac{(E_6)_x}{Sp(4)_x} \ , \ \frac{(E_7)_x}{SU(8)_x} \ , \ \frac{(E_8)_x}{SO(16)_x} \tag{8.6b}$$

with maximal dimension across simply-laced g. The K-conjugate solution (8.4c) is therefore the set of corresponding coset constructions for g/h.

The second K-conjugate pair, which we call simply-laced $g^\#$,

$$\lambda = \frac{1}{2\tilde{h}(\tilde{h}+x)}[1 - \eta B^{-1}(2\tilde{h}^2 + \tilde{h}(4-x) - 2x^2 + 10x - 16)] \tag{8.7a}$$

$$L_- = \frac{1}{2(\tilde{h}+x)}[1 + \eta B^{-1}(x\tilde{h} - 6x + 16)] \ , \ L_+ = -\eta B^{-1}(x-4) \tag{8.7b}$$

$$c = \frac{x\, rank\, g}{2(\tilde{h}+x)}[\tilde{h} + 1 + \eta B^{-1}(\tilde{h}^2(x-2) + \tilde{h}(12-5x) + 2x^2 - 10x + 16)] \tag{8.7c}$$

$$B \equiv \sqrt{\tilde{h}^2 x^2 + 4\tilde{h}(x^3 - 13x^2 + 40x - 32) + 4(x^4 - 10x^3 + 41x^2 - 80x + 64)} \tag{8.7d}$$

($\eta = \pm 1$) is unitary with generically-irrational central charge across all levels of simply-laced g.

The operator form of the construction

$$L = \rho^{-2} \left(\lambda \tilde{h} \sum_A T_{AA} + \frac{1}{2}(L_+ - L_-) \sum_{\rho > 0} {}^*_*(E_\rho - E_{-\rho})^2 {}^*_* + \right.$$

$$\left. + \frac{1}{2}(L_+ + L_-) \sum_{\rho > 0} {}^*_*(E_\rho + E_{-\rho})^2 {}^*_* \right) \tag{8.8}$$

shows complete symmetry among the roots of g, according to the ansatz, but asymmetry among the h and $Cartan\, g + R = g/h$ components of the symmetric space. Similar structure on symmetric spaces was seen in the spin-orbit constructions [15].

Simply-laced $g^{\#}$ is rational for $SU(2)_x$, $SO(4)_x$ and also for $x = 1, 2, 4$ and for $\tilde{h} = 2n$, $x = n+3$. All these points may be identified with known h and g/h as above. The lowest irrational central charges of the construction are found at level three,

$$c(SU(3)^{\#}_3) = 2\left(1 - \frac{1}{\sqrt{73}}\right) \simeq 1.7659 \tag{8.9a}$$

$$c(SO(6)^{\#}_3) = \frac{45}{14}\left(1 - \frac{1}{5\sqrt{2}}\right) \simeq 2.7597 \tag{8.9b}$$

$$c((E_6)^{\#}_3) = \frac{39}{5}\left(1 - \frac{14}{13\sqrt{19}}\right) \simeq 5.8729 \tag{8.9c}$$

$$c((E_7)^{\#}_3) = \frac{19}{2}\left(1 - \frac{137}{19\sqrt{697}}\right) \simeq 6.9054 \tag{8.9d}$$

$$c((E_8)^{\#}_3) = \frac{124}{11}\left(1 - \frac{407}{31\sqrt{1969}}\right) \simeq 7.9374 \tag{8.9e}$$

and the central charge of $SU(3)^{\#}_3$ in (8.9a) is the lowest irrational value we have found. More generally, the central charges in (8.7c) increase irrationally with x and $rank\, g$ at fixed η, and approach correctly-bounded integers at high level

$$\lim_{k \to \infty} c = \dim \Phi_+(g) + \frac{1+\eta}{2} \mathrm{rank}\, g \tag{8.10}$$

as they should according to (4.5b).

We also remark on the irrational conformal weights $\Delta = c/6x$ of $SU(3)^{\#}_x$, computed with (2.13) for the 3 or $\bar{3}$ of $SU(3)_x$; these weights apparently lie in a general family of one fermion conformal weights $\Delta(one\ fermion) = c/2c_{(free\ fermions)}$ derived for fermionic Sugawara constructions in [12].

This completes the study of the maximally-symmetric subansatz (8.2) for simply-laced g in (3.7). More general subansätze are obtained as follows. A convenient basis in (3.7) is $L^{AB} = \sum_{i,j} \lambda_{ij} \alpha_i^A \alpha_j^B$ with $\{\alpha_i,\ i = 1, 2, \cdots, rank\, g\}$ the simple roots of simply-laced g. General analysis of (3.7) proceeds with the Cartan matrix of g. Subansätze with the symmetry of the Dynkin diagram of g are automatically consistent: as an example, the $\alpha \leftrightarrow \beta$ symmetric subansatz for $g = SU(3)$ in (3.7),

$$L^{AB} = \lambda_\alpha(\alpha^A\alpha^B + \beta^A\beta^B) + \lambda_{\alpha\beta}\alpha^{(A}\beta^{B)} \quad , \quad \alpha, \beta > 0 \tag{8.11a}$$

$$L^{\beta,\beta} = (L^{\alpha,\alpha})^* \ , \ L^{\beta,-\beta} = L^{\alpha,-\alpha} \ , \ L^{\gamma,\pm\gamma} = real \ , \ \alpha + \beta = \gamma \tag{8.11b}$$

gives five coupled quadratic equations and a linear equation when $L^{\rho\rho} \neq 0$.

9 Discussion

Our excursion into affine-Virasoro space has been successful. The master equation is so far analytically tractable, and there is every indication that the growing list of irreducible analytic unitary constructions

$$\begin{gathered}\text{Cartan}\, g^{\#};\ \ SU(2)_4^{\#};\ \ (SU(2)_x \times SU(2)_x)^{\#},\ x \neq 4\ ;\ .... \\ (SU(2)_x)_{\#}^{q\geq 3};\ \text{simply-laced}\ g^{\#}\ ;\ ....\end{gathered} \tag{9.1}$$

is only the beginning of a much larger picture.

We are impressed with the abundance of unitary constructions in these ansätze, which augurs the possible dominance of irrational unitary over rational unitary solutions on compact g: the quadratic deformations are non-unitary outside their indicated ranges and four discrete non-unitary points were found on $(SU(2))^3$. Indeed, the pair of constructions on $(SU(2)_3)^3$ with $c = (1 - i\eta/\sqrt{39})27/10$ is the only non-unitary irrational construction encountered.

Our work here was primarily within the operator ansatz $\{T_{AB}, T_{\rho,\pm\rho}\}$ of (3.7), and we studied only maximally-symmetric subansätze for the larger groups. The $\{T_{AB}, T_{\alpha\beta}\}$ ansatz (3.8) for $SU(2)^q \times U(1)^p$ remains essentially unstudied, and we have indicated subansätze with lower symmetry within $\{T_{AB}, T_{\rho,\pm\rho}\}$ for future study. It is also likely that other consistent ansätze can be found, involving, for example, the operators $T_{A\alpha}$ which appear in the spin-orbit construction.

All new solutions of the master equation, including the generalized spin-orbit construction, involve no more than a single square root, although every case began with large systems of coupled quadratic equations. This surprise indicates a deep structure in the master equation which we do not understand beyond the following phenomenological remarks:

1. K-conjugation invariance on a given manifold G effectively reduces a $(2n)^{th}$-order algebraic equation to n^{th} order, as seen in

$$0 = \prod_{a=1}^{n}(x - x_a)(x - \tilde{x}_a) = \prod_{a=1}^{n}((x - \frac{1}{2}x_g)^2 - (x_a - \frac{1}{2}x_g)^2) \tag{9.2}$$

   when $x_a + \tilde{x}_a = x_g$. As a result, subsystems of up to three coupled quadratic equations can be solved analytically.

2. The role of K-conjugation from submanifolds of G is more important. The high order of the equations on a given manifold G, and the exponential growth of $N_K(g)$ in (2.11), is partially explained by affine-Virasoro nests from submanifolds $H \subset G$. The deficit on G is the set of irreducible solutions $L_g^{\#}$ on G, which are obscured by the nesting from below. The phenomenon is that nesting has so far provided enough rational solutions, and other known solutions, on each manifold to further factor the ansätze down to a single quadratic equation for each new $L_g^{\#}$.

The existence of these new constructions opens a variety of other directions. One direction is to find the *affine-Virasoro master action* for all solutions to the master equation, generalizing [30] for the simple coset constructions. Another direction involves alternate routes to further $c > 1$ theories, such as generalized Feigin-Fuchs constructions [11]. An idea here is to begin with infinite-dimensional unitary representations of non-compact groups with complex spin (e.g. the $j = -\frac{1}{2} + i\lambda$ representations of SL(2,R)) and raise c by adjusting imaginary c-changing deformation parameters to obtain real conformal weights.

Appendix: Quadratic Deformation Operators and (1,0) States

A general quadratic conformal deformation is a continuous family of affine-Virasoro operators $L^{ab}(\lambda)\, {}^*_* J_a J_b {}^*_*$ whose coefficients solve the Virasoro master equation [15, 17]

$$2L^{ab}(\lambda) = L^{cd}(\lambda)L^{ef}(\lambda)R_{cdef}{}^{ab}\ ,\ \ c(\lambda) = 2G_{ab}L^{ab}(\lambda) \tag{A.1}$$

with λ a set of continuous parameters. The K-conjugate partner $\tilde{L}$ of L is well-defined by

$$L(\lambda) + \tilde{L}(\lambda) = L_g\ \ ,\ \ c(\lambda) + \tilde{c}(\lambda) = c_g \tag{A.2}$$

since the Sugawara construction L_g is not quadratically deformable by the currents of g [15].

Any solution of the linearized master equation[††] about the point λ

$$\delta^{ab}(\lambda) = \delta^{cd}(\lambda)L^{ef}(\lambda)R_{cdef}{}^{ab} \tag{A.3}$$

defines a *quadratic deformation operator* $\delta^{ab}(\lambda)\, {}^*_* J_a J_b {}^*_*$ of $L(\lambda)$ such that

$$L(\lambda+\delta\lambda) \simeq L(\lambda) + \delta(\lambda)\ ,\ \ c(\lambda+\delta\lambda) \simeq c(\lambda) + \delta c(\lambda)\ ,\ \ \delta c(\lambda) = 2G_{ab}\delta^{ab}(\lambda) \tag{A.4}$$

is also Virasoro in a neighborhood of λ. Moreover, according to (A.2), the operator $-\delta(\lambda)$ is automatically a deformation operator of $\tilde{L}(\lambda)$ with $\delta\tilde{c}(\lambda) = -\delta c(\lambda)$. The commutator of the Virasoro operator $L^{(m)}(\lambda)$ with its deformation operator $\delta^{(n)}(\lambda)$

$$\begin{aligned}[L^{(m)}(\lambda), \delta^{(n)}(\lambda)] =& \frac{1}{24}\delta c(\lambda) m(m^2-1)\delta_{m+n,0} + q(m,n)M^a(\lambda)J_a^{(m+n)} \\ &+ \frac{1}{2}(m-n)\delta^{(m+n)}(\lambda) + N^{abc}(\lambda)W_{abc}^{(m+n)}\end{aligned} \tag{A.5a}$$

$$M^e(\lambda) = L^{ab}(\lambda)\delta^{cd}(\lambda)Q_{abcd}{}^e\ ,\ \ N^{efg}(\lambda) = L^{ab}(\lambda)\delta^{cd}(\lambda)S_{abcd}{}^{efg}, \tag{A.5b}$$

$$q(m,n) = \frac{1}{6}(m+n+1)(m+n+2) - \frac{1}{2}(m+1)(n+1), \tag{A.5c}$$

follows with (A.1) and the OPE of [15], where the tensors Q, S and the three-current operator W_{abc} are also defined.

It is instructive to translate this commutator onto the state

$$|\delta(\lambda)\rangle = \delta(0,\lambda)|0\rangle = \delta^{(-2)}(\lambda)|0\rangle,\ \ \delta(z,\lambda) = \sum_m \delta^{(m)}(\lambda)z^{-m-2} \tag{A.6}$$

[††]Solutions of the linearized master equation (A.3) correspond to metric deformations δg_{ij} (about a metric g_{ij}) which solve the linearized Einstein equation $\delta\hat{R}_{ij} + \delta g_{ij} = 0$ in the geometric formulation [16].

created by the deformation operator on the $SL(2,R)$-invariant vacuum. The result

$$L^{(0)}(\lambda)|\delta(\lambda)\rangle = |\delta(\lambda)\rangle + \frac{1}{2}M^a(\lambda)J_a^{(-2)}|0\rangle \tag{A.7a}$$

$$L^{(1)}(\lambda)|\delta(\lambda)\rangle = M^a(\lambda)J_a^{(-1)}|0\rangle \tag{A.7b}$$

$$L^{(2)}(\lambda)|\delta(\lambda)\rangle = \frac{1}{4}\delta c(\lambda)|0\rangle \ , \ L^{m>2}(\lambda)|\delta(\lambda)\rangle = 0 \tag{A.7c}$$

is obtained with (A.5)-(A.6) and the vacuum structure (2.6). No c-changing quadratic deformations ($\delta c \neq 0$) have yet been found, and, in fact, all the deformation examples (Cartan $g^\#$, $SU(2)_4^\#$ and $(SU(2)_x \times SU(2)_x)^\#$, $x \neq 4$) of the text realize only a particularly interesting special case of the generic relations (A.5a) and (A.7).

Using the explicit form of $L(\lambda)$, $\delta(\lambda)$ in each case, we have verified that

$$\delta c(\lambda) = M^a(\lambda) = 0 \ \Leftrightarrow \ [L^{m\geq 0}(\lambda) - \delta_{m,0}]|\delta(\lambda)\rangle = 0 \tag{A.8}$$

in all the examples, so that $|\delta(\lambda)\rangle$ is a (1,0) Virasoro highest weight state of $L(\lambda)$ throughout each of the known quadratic conformal deformations.

A simple example of this phenomenon is easily checked as follows: The endpoint theories $\lambda\alpha^4 = 1/4$, $-1/12$ of $SU(2)_4^\#$ are $L = T_{33}/2k$ and $\tilde{L} = L(SU(2)_x) - L$ at $x = 4$, where the deformation operator $\delta = \partial L^\# / \partial\lambda$ reduces to $\delta \sim T_{\alpha\alpha}$. The relation (4.3b) of the text shows that the state $T_{\alpha\alpha}^{(-2)}|0\rangle$ is Virasoro highest weight of L and $\tilde{L}$ with

$$\Delta_{\alpha\alpha} = 2 - \tilde{\Delta}_{\alpha\alpha} = \frac{4}{x} \tag{A.9}$$

for any level x. It follows that $T_{\alpha\alpha}^{(-2)}|0\rangle$ is a (1,0) highest weight state of both L and $\tilde{L}$ when $x = 4$, in accord with (A.8).

Conversely, states satisfying (A.7) or (A.8) correspond directly to deformation operators satisfying (A.5a) or

$$[L^{(m)}(\lambda), \delta^{(n)}(\lambda)] = \frac{1}{2}(m-n)\delta^{(m+n)}(\lambda) + N^{abc}(\lambda)W_{abc}^{(m+n)} \tag{A.10}$$

when $\delta^{(-2)}(\lambda)|0\rangle$ is (1,0) across the deformation. The W-term in (A.10) is not generally zero. According to (A.10), the deformation operators which create the (1,0) states are not (1,0) primary fields of $L(\lambda)$, the term "$\frac{1}{2}(2,0)$ operators" being more descriptive.

The (1,0) phenomenon (A.8) further motivates physical comparison of the quadratic deformation examples with particular c-fixed $SL(2,R)$-preserving linear deformations [11], where the analogous (1,0) states across the deformation are $D^A J_A^{(-1)}|0\rangle$, $A \in$ Cartan g.

Acknowledgements

We thank O. Alvarez, K. Bardakci, R. Cahn, E. Rabinovici and M. Shifman for helpful discussions.

References

[1] V.G. Kač, Funct. Anal. App. 1 (1967) 328; R.V. Moody, Bull. Am. Math. Soc. 73 (1967) 217.

[2] K. Bardakci and M.B. Halpern, Phys. Rev. D3 (1971) 2493.

[3] P. Ramond, Phys. Rev. D3 (1971) 2415.

[4] M.B. Halpern, Phys. Rev. D4 (1971) 2398.

[5] M.B. Halpern, Phys. Rev. D12 (1975) 1684.

[6] T. Banks, D. Horn and H. Neuberger, Nucl. Phys. B108 (1976) 119.

[7] I. B. Frenkel and V.G. Kač, Inv. Math. 62 (1980) 23; G. Segal, Comm. Math. Phys. 80 (1981) 301.

[8] E. Witten, Comm. Math. Phys. 92 (1984) 455; G. Segal, unpublished; V.G. Knizhnik and A.M. Zamolodchikov, Nucl. Phys. B247 (1984) 83.

[9] P. Goddard, A. Kent and D. Olive, Phys. Lett. 152B (1985) 88.

[10] D. Gross, J.A. Harvey, E. Martinec and R. Rohm, Phys. Rev. Lett. 54 (1985) 502.

[11] J. K. Freericks and M.B. Halpern, Ann. of Phys. 188 (1988) 258; Erratum, ibid. 190 (1989) 212.

[12] M.B. Halpern, preprint UCB-PTH-88/30. To appear in Ann. of Phys.

[13] P. Goddard and D. Olive, Int. J. Mod. Phys. A1 (1986) 303.

[14] I.B. Frenkel, J. Lepowsky and A. Meurman, to appear.

[15] M.B. Halpern and E. Kiritsis, Mod. Phys. Lett. A4 (1989) 1373.

[16] M.B. Halpern and J.P. Yamron, preprint IASSNS-HEP-89/23.

[17] A.Yu. Morozov, A.M. Perelomov, A.A. Rosly, M.A. Schifman and A.V. Turbiner, preprint CTP#1741 (1989).

[18] E. Kiritsis, Mod. Phys. Lett. A4 (1989) 437; G. V. Dunne, I. G. Halliday and P. Suranyi, Phys. Lett. B213 (1988) 139.

[19] A.A. Beilinson and V.V. Schechtman, Comm. Math. Phys. 118 (1988) 651; N. Sakai and P. Suranyi, Nucl. Phys. B318 (1989) 655; M. Bershadsky and H. Ooguri, preprint IASSNS-HEP-89/9 (1989); M. Evans and B.A. Ovrut, preprint UPR-0392 (1989).

[20] K.S. Narain, Phys. Lett. 169B (1986) 41; K.S. Narain, M.H. Sarmadi and E. Witten, Nucl. Phys. B279 (1987) 369.

[21] A. Schwimmer and N. Seiberg, Phys. Lett. 184B (1987) 191.

[22] S. Ferrara, C. Kounnas and M. Porrati, Nucl. Phys. B304 (1988) 500; I. Antoniadis, C. Bachas and C. Kounnas, Phys. Lett. 200B (1988) 297.

[23] M.B. Halpern and C.B. Thorn, Phys. Rev. D4 (1971) 3084; E. Corrigan and D.B. Fairlie, Nucl. Phys. B91 (1975) 527; W. Siegel, Nucl. Phys. B109 (1976) 244.

[24] L. Dixon, J. A. Harvey, C. Vafa and E. Witten, Nucl. Phys. B261 (1985) 620; ibid. B274 (1986) 285.

[25] D.B. Fairlie as quoted in A. Chodos and C.B. Thorn, Nucl. Phys. B72 (1974) 509; J. -L. Gervais and A. Neveu, Nucl. Phys. B224 (1983) 329; B.L. Feigin and D.B. Fuchs, preprint Moscow University (1983); V.L. Dotsenko and V. Fateev, Nucl. Phys. B240 (1984) 312; D. Friedan, E. Martinec and S. Shenker, Nucl. Phys. B271 (1986) 93.

[26] S. Mandelstam, Phys. Rev. D7 (1973) 3777.

[27] E. Witten, preprint IASSNS-HEP-89/38.

[28] L. Dixon, M.E. Peskin and J. Lykken, preprint SLAC-PUB-4884 (1989); I. Bars, preprint USC-89/HEP 02 .

[29] I.B. Frenkel, J. Funct. Anal. 44 (1981) 259.

[30] W. Nahm, Duke Math. J. 54 (1987) 579; K. Bardakci, E. Rabinovici, and B. Saering, Nucl. Phys. B299 (1988) 151; D. Altschuler, K. Bardakci, and E. Rabinovici, Comm. Math. Phys. 118 (1988) 241; D. Karabali, Q.-H. Park, H.J. Schnitzer, and Z. Yang, Phys. Lett. 216B (1989) 307; K. Gawedzki and A. Kupiainen, Nucl. Phys. 320B (1989) 625.