Analytic Aspects of Rational Conformal Field Theories[*]

Elias B. Kiritsis

Department of Physics[†]
University of California
and
Theoretical Physics Group
Physics Division
Lawrence Berkeley Laboratory
1 Cyclotron Road
Berkeley, California 94720

Abstract

The problem of deriving linear differential equations for correlation functions of Rational Conformal Field Theories is considered. Techniques from the theory of Fuchsian differential equations are used to show that knowledge of the central charge, dimensions of primary fields and fusion rules are enough to fix the differential equations for one and two-point functions on the torus. Any other correlation function can be calculated along similar lines. The results settle the issue of "exact solution" of Rational Conformal Field Theories.

[†]email address: KIRITSIS@LBL.bitnet

1. INTRODUCTION

Following the pioneering work of BPZ, [1], and the recent interest in string theories as theories unifying the known forces of nature there has been considerable activity in trying to understand two-dimensional Conformal Field Theory (CFT), which is the underline building block of string theories and describes two-dimensional critical statistical systems.

The big advantage of the program of CFT is that it reduces the problem of solving a continuum field theory to an almost algebraic problem, that of analysing the representations of the two-dimensional Conformal Group and of finding ways to tie them together into a sensible quantum field theory. The use of representations of the Conformal Group bypasses a by now intractable problem in higher dimensions, that of finding a classification scheme, (a basis) for the operators of a Quantum Field Theory.

There are two major problems in the context of CFT. The first is classifying all possible CFT's. The second is to solve them exactly. So far both problems escape solution and it seems that a more mature understanding of the framework of CFT is needed along with some probably new mathematics. Attempts so far in these directions focused into considering subclasses of CFT's that present simplifying features. Such classes include G/H models and Rational CFT's. There are some partial classifications at hand like that of unitary theories with $c < 1$, [2], or RCFT's with $c = 1$, [3]

There is a variety of mathematical tools used so far, more or less successfully, to deal with the aforementioned problems. Our concept of a CFT was considerably advanced by the nice work of [4] which defined CFT in terms of vector bundles on moduli spaces.* Despite the fact that such an approach focuses on objects that are "marginal" in traditional treatments of Quantum Field Theory, it is very suitable to the use of powerful mathematical tools in order to study the CFT.

Most of the concrete results that exist so far pertain to RCFT's. For us an operational definition of RCFT is a CFT with the property that any correlation function has a finite number of blocks. There has been a lot of approaches to the classification problem of RCFT's. Most of them are algebraic in character and use as basic quantities the monodromy matrices of various correlation functions, [6-11]. The analytic problem has been mainly

---

*A more rigorous approach based on the same idea was developed later in [5].

ignored until recently, with a few exceptions, [12]. However in the last year it has been recognized that differential equation techniques may be in fact a very powerful tool in both the classification and especially in the exact solution of RCFT's, [13-22]. The existence of such linear differential equations for correlation functions was recognized in the pioneering work of BPZ. They were derived from the null vectors of the Virasoro algebra. RCFT's are always "minimal models" of some extended chiral algebra and one would expect differential equations coming from the null vectors of that algebra. In fact the picture of CFT in terms of flat vector bundles guarantees the existence of such differential equations. Such differential equations can, not only be used to solve the theory but for classification purposes as well, [16,18].

In some previous work, [18], we applied the machinery of Fuchsian Differential Equations, (FDE), to study the classification problem for characters on the torus. In this paper we are going to focus on other objects of CFT. Some basic general techniques in this direction were introduced in [16]. Here we are going to give a complete description of procedures that allow one to determine completely differential equations for any correlation function of RCFT.

Our results are as follows. We first describe how one can determine differential equations for one-point functions on the torus. Information for such one-point functions is crucial in determining differential equations for two-point functions on the torus which is a basic object in CFT. We prove that knowledge of the value of the central charge, dimensions of the primary fields and fusion rules is enough to determine differential equations for any correlation function of the theory. The crucial step is to be able to deal successfully with the two point function on the torus. Its determination fixes at the same time the structure constants of the theory as well as basic algebraic objects as the braiding and fusion matrices. Basic tools include the theory of FDE's and the associated Riemann-Hilbert problem as well as the theory of isomonodromic deformations and function theory on the appropriate surface. We would like at this point to apologize in advance for the concerned reader since at several points we preferred to sacrifice mathematical rigor in favour of clarity(?).

The structure of the present paper is as follows. In section 2 we give a semi-intuitive discussion of the emergence of (projectively) flat vector bundles in RCFT and how one can, formally, from the knowledge of the bundle structure derive differential equations for the sections. We follow the ap-

2

proach of [23] based on his solution of the R-H problem on analytic varieties. In section 3 we deal extensively with differential equations for one-point functions on the torus. We show how the techniques developed for characters, [18], can be applied also for one-point functions and we discuss certain issues that arise, in particular the subject of apparent singularities. In section 4 we focus on the two-point functions on the torus. We show how we can determine the appropriate differential equation by using information about one-point functions on the torus as well as its degeneration properties when the torus degenerates to the four-punctured sphere. In particular we show that the reducibility of the monodromy representation of the equation on the sphere provides with enough constraints that fix the differential equation uniquely. In section 5 we discuss the case of other correlation functions. We present techniques based on isomonodromic deformations which are very helpful in determining differential equations for higher-point functions. Section 6 contains our concluding remarks. In appendix A we discuss some explicit examples of degeneration of the differential equation for two-point functions on the torus. In appendix B we present the proof that factorization of the monodromy group on the sphere provides enough conditions on the differential equation for the two-point functions on the torus. In appendix C we discuss an example of determining a two-point function on the torus when the two operators are not conjugate. Finally in appendix D we discuss a calculational technique for solving the equation for two-point functions on the torus by mapping them on the branched sphere and then using standard contour-integral representations.

2. FLAT VECTOR BUNDLES ON RIEMANN SURFACES AND LINEAR DIFFERENTIAL EQUATIONS.

Friedan and Shenker, [4], provided a description of Conformal Field Theory (CFT) in terms of flat vector bundles[‡] on Riemann surfaces and their factorization algebra. The use of vector bundles rests on the by now familiar concept that if one knows the value of the central charge, critical exponents and how conformal blocks transform under the appropriate modular group of the relevant surface with punctures then this is enough to define the theory.

[‡]In fact the appropriate bundles are projectively flat due to the conformal anomaly. In [4] it was shown how this can be taken into account. ¿From now on we will omit the term "projective" for simplicity. We will come back to it when appropriate.

Consider a correlation function on a given Riemann Surface. If we choose a homology basis on the surface we can define the holomorphic and antiholomorphic conformal blocks as the contributions of the various channels specified by the chosen homology basis. The conformal blocks depend analytically on the moduli of the punctured surface. A different choice of homology basis would give a different choice of conformal blocks. The requirement is that the physical correlation function is independent of the choice of the homology basis.[§] The correlation function should also transform appropriately under global diffeomorphisms of the surface.

The concept of the vector bundle arises from the previous remarks. Thus the base space is the Teichmüller space of the punctured Riemann surface and the fiber over a particular point is a vector space generated by the conformal blocks at that point. The vector bundle is holomorphic since the conformal blocks vary holomorphically over moduli space. The group of the bundle is generically $GL(n, C)$ where n is the dimensionality of the fibers (the number of conformal blocks[¶]). The action of the modular group of the punctured surface on the fibers defines a way of transport in the bundle space and hence a connection. This connection is locally flat (the associated curvature is zero). Consequently the modular group defines a homeomorphism (representation) from the fundamental group of the base space to $GL(n, \mathbf{C})$. When this representation is irreducible then n is the rank of the vector bundle. The conformal blocks are sections of the flat vector bundle described above.

One of the questions that are of central interest in this paper, is the following: To what extend, knowledge of the global data specifying the vector bundles in question is enough to determine the sections analytically? This in standard language would be called "exact solution" of the respective CFT. Associated to the previous question is the following: What is the minimum amount of data enough to specify the solution of the theory?

The approach here would be to use the global data of a specific vector bundle in order to determine a Linear Differential Equation of the Fuchsian type (FDE), whose solutions form a basis for the sections of the bundle and are thus the building blocks of the appropriate physical correlation function. We will now sketch the general procedure, due to Deligne, [23], which, given

---

[§]This property usually comes under the term duality and it equivalent to the associativity of the operator algebra of the CFT.

[¶]Since in this paper we are focusing on Rational CFT's, this number is finite by definition.

a local system of vector spaces with prescribed global transformation properties determines a FDE whose solutions are the holomorphic sections of the associated vector bundle.

Let M be a for concreteness a compact Riemann surface and S a set of m points on M. Consider the homeomorphism $\rho : \pi_1(M - S) \to GL(n, \mathbf{C})$. We will determine a FDE with "monodromy group" isomorphic to ρ. This can be done in the following steps, [23],

(1) Take a local system $\tilde{V}$ of n-dimensional vector spaces on $M - S$ associated to the representation ρ.

(2) The local system $\tilde{V}$ determines in a canonical way a holomorphic vector bundle $\tilde{U}$ on $M - S$ with a flat connection, $\tilde{\nabla}$ so that $\tilde{V}$ is generated by the horizontal (covariantly constant) sections of $\tilde{U}$, $\tilde{V} = [\xi \in \tilde{U} : \tilde{\nabla}\xi = 0]$.

(3) Extend the pair $(\tilde{U}, \tilde{\nabla})$ to a pair (U, ∇) onto the whole of M, where U is a holomorphic vector bundle on M and ∇ is a meromorphic connection on M with simple poles on S.

(4) To derive a Linear Differential Equation, take a holomorphic section, ϕ of the dual bundle, U^*. Consider the local system $\phi(\tilde{V})$ of meromorphic functions as a sub-sheaf of $\mathcal{O}_{M-S}$. Then $\phi(\tilde{V})$ is isomorphic to $\tilde{V}$ and the differential equation with solution sheaf $\phi(\tilde{V})$ is the desired one. This equation is of the Fuchsian type since the connection ∇ on M has at most single poles. The procedure above can be always carried out provided one allows for extra "apparent" singularities in the FDE, (see [24]).

3. CHARACTERS AND ONE-POINT FUNCTIONS ON THE TORUS.

In reference [16] a classification procedure was proposed for characters of RCFT's relying on linear differential equations. In [18] advantage was taken from the fact that such equations are of the Fuchsian type to constrain them considerably by using standard results on the solution of the Riemann-Hilbert problem in the case of one variable. In this section we are going to expand on reference [18] since this seems to be necessary in the subsequent discussion of one-point functions on the torus, a subject that has only been touched upon in [18].

Information about one-point functions is crucial in determining gross features of two-point functions on the torus. This is certainly apparent from reference [16] where a procedure for determining differential equations for

two-point functions was developed. As we shall see the arguments presented there will turn out not to be enough in the general case and the purpose of the next section is to clarify what is really needed in order to be able to write down a differential equation for any two-point function in a RCFT provided we know its characters.

So let's turn our attention first to characters of RCFT. Let $f_i(q)$, $i = 1, ..., n$ be the n characters of a RCFT[‖]. They are functions of the modulus of the torus and transform under a finite-dimensional representation of the modular group of the torus, Γ. The action of Γ restricts the characters to the fundamental region of Γ which is a compact Riemann surface of genus zero with three orbifold singularities at $\tau = i, e^{2\pi i/3}, i\infty$ of respective orders $2, 3, \infty$. In order to avoid complications arising from the finite order orbifold points it is more convenient to work on the fundamental region of $\Gamma(2) \subset \Gamma$ which is a six-fold cover of ordinary moduli space and it is analytically isomorphic to the three-punctured sphere. We choose the analytic coordinate on the punctured sphere to be,

$$x = \left[\frac{\vartheta_2(\tau)}{\vartheta_3(\tau)} \right]^4 \tag{3.1}$$

in which case the three punctures (corresponding to $\tau = i\infty$) are at $x = 0, 1, \infty$, the second order orbifold point, $\tau = i$ is mapped to $x = -1, 2, 1/2$ and the the third order orbifold point is mapped to $x = e^{\pm i\pi/3} \equiv \rho$.

The general arguments presented in the previous section imply that the characters, f_i, satisfy an n-th order differential equation in x with coefficient functions being rational functions of x. This equation should be of the Fuchsian type with regular singularities at $0, 1, \infty$. It should also be modular invariant, that is form invariant under $S : x \to 1 - x$ and $T : x \to \frac{x}{x-1}$. Standard arguments in RCFT[**] imply that the solutions should not contain logarithmic singularities anywhere on the sphere and they should not have poles or branch cuts away from the punctures.

Such an equation in general will contain "apparent" singularities when the Wronskian has zeros away from the punctures. A singularity is called "apparent" when the monodromy around it, is trivial, [24]. This is achieved

---

[‖]Our notation and conventions are those of [18].
[**]See for example [18]

when all the indices of that singularity are non-negative integers and loga-rithmic singularities around that point are absent. The presence of apparent singularities is generically important to provide *always* a solution to the Riemann-Hilbert problem: Given a monodromy representation, $\rho : \pi_1(S_p) \to GL(n, \mathbf{C})$, to find a FDE realizing that representation[††], S_p being the three-punctured sphere. The reasoning for the above can be based on simple pa-rameter counting, [24]. An important point used in [18] was that for an n-th order FDE the number of apparent singularities is bounded from above, [25]. To be more specific an n-th order FDE with three regular singularities must have at most $(n-1)(n-2)/2$ apparent singularities. This fact along with modular invariance which implies that apparent singularities come in groups of two, three or (generically) six, poses important constraints on the relevant equations.

By taking into account all the above we can write the most general such n-th order FDE as,

$$D^n f + \sum_{i=1}^{n} p_i(x) D^i f = 0 \tag{3.2}$$

where the D^i are differential operators that are defined in appendix A of [18] and

$$p_k(x) \sim \frac{[Q_\rho^2(x) Q_i(x)]^k \prod_{i=1}^{k(N-1)} Q_{b_i}(x)}{[x(x-1) \prod_{i=1}^{N} Q_{a_i}(x)]^k} \tag{3.3}$$

with,

$$Q_a(x) \equiv (x-a)(x-\frac{1}{a})(x-1+a)(x-\frac{1}{1-a})(x+\frac{a}{1-a})(x+\frac{1-a}{a}) \tag{3.4.a}$$

$$Q_\rho(x) \equiv x^2 - x + 1 \ , \ Q_i(x) = (x+1)(x-2)(x-\frac{1}{2}) \tag{3.4b}$$

The number N in (3.3) is bounded above according to the discussion of the previous paragraph. For example when $n = 2, 3$ then $N = 0$, when $n = 4$ then $N \leq 2$ and potential apparent singularities must be only at the images of the orbifold points and so on.

The indices around $x = 0$ are related to the central charge, c and the dimensions of the primary fields, Δ_i, of the maximal chiral algebra of the

---

[††]By construction here, a monodromy representation is equivalent to a representation of the modular group of the torus, Γ.

theory. If we assume that the dimensions are non-negative then the smaller index should be equal to $-\frac{c}{12}$, (corresponding to the character of the identity representation, f_0) and the rest should be equal to $-\frac{c}{12} + 2\Delta_i$. The indices around $x = 1, \infty$ are related to those around $x = 0$ by the modular invariance of the equation. The Fuchs relation[‡‡] is, [16],

$$2n(n-1) + nc - 24\sum_{i=1}^{n-1}\Delta_i = 4l \tag{3.5}$$

where n is the order of the equation and l is a non-negative integer, ($l \neq 1$). $l/6$ is the number of zeros of the Wronskian in moduli space, where a zero at the third order ramification point counts as $1/3$, at the second order point $1/2$ and is integer elsewhere inside moduli space.

There are of course some extra severe constraints on the solutions of the FDE. The character of the identity, f_0, when normalized so that there is a single identity operator, should have integer Fourier coefficients. Explicitly,

$$f_0(q) = q^{-\frac{c}{24}}\left(1 + \sum_{n=1}^{\infty} a_n q^n\right) \tag{3.6}$$

with a_n integers. If one wants a unitary theory then it is necessary (but not sufficient) that all Fourier coefficients be non-negative integers. In general the normalization of characters corresponding to representations other than the identity is not obvious. This however can be done, [16,18,22], using modular invariance. There are also extra constrains. One can calculate the action of the modular transformation S on the characters which is nothing else but the monodromy matrix S_{ij} of the transformation $x \to 1 - x$. Then using the Verlinde formula the number of couplings N_{ijk} of three representations i, j, k ("Fusion rules") can be calculated and they should be non-negative integers. There are examples of solutions which pass all the previous tests but fail the last, [16].

¿From now on we will assume that we know a collection of characters calculated along the lines specified above. In particular all we need to know is c, Δ_i and the fusion rules, N_{ijk}.

Let's now focus our attention on un-normalized one-point functions on the torus. Since translation invariance is still a good symmetry of the torus

---

[‡‡]It is equivalent in this case to the Riemann-Roch theorem for the Wronskian in the moduli space of the torus.

such one-point functions are independent of the position of the operator in question. They only depend on the modulus of the torus. For concreteness consider the one point function of an operator $\Phi(z, \bar{z})$ of dimension $(\Delta, \bar{\Delta})$. We will be considering the holomorphic blocks of this correlation function. The number of blocks is uniquely specified by the fusion rules. Call $g_i(q)$ the block corresponding to the i-th representation going around in the loop. If Φ belongs to the j-th representation then the representation i going around the loop contributes only if $N_{iij} \neq 0$. Thus as $q \rightarrow 0$ the blocks behave as,

$$g_i(q) \sim q^{-\frac{c}{24}+\Delta_i}[1 + \mathcal{O}(q)] \tag{3.7}$$

Under modular transformations the blocks transform among themselves. However since a general modular transformation involves a change of scale the modular transformation matrices are τ-dependent. Another way to put this is that since the operator Φ has non-zero dimension the g_i transform as forms of weight Δ. Thus if we define new "blocks" by $f_i(\tau) \equiv g_i(\tau)\eta^{-2\Delta}(\tau)$, where η is the standard Dedekind η-function then the f_i transform under a modular transformation A with weight zero, that is like characters,[*]

$$f_i(A\tau) = (M_A)_{ij}f_j(\tau) \tag{3.8}$$

where M_A is τ-independent. The behaviour of f_i as $q \rightarrow 0$ now becomes,

$$f_i(q) \sim q^{-\frac{c+2\Delta}{24}+\Delta_i}[1 + \mathcal{O}(q)] \tag{3.9}$$

Thus the f_i transform exactly like characters and their indices around $x = 0$ can be read off (3.9). The whole machinery of differential equations that was applied to characters can, at no extra cost, be applied to the redefined blocks of the one-point functions.

Let's indicate some points in special cases before we embark in general statements. We would like to study the one-point functions of descendants of the identity operator in theories with two blocks. Such theories were classified in [18]. The dimension of the non-trivial representation is related to the central charge by $\Delta = \frac{c+2}{12}$. Consider a descendant field of the identity at level $N \geq 0$. Its dimension is by definition N. The number of blocks of its one-point function is the same as the number of characters, namely two.

---

[*]Note that since the η-function has no zeros or poles inside moduli space, it only changes the order of the poles at infinity leaving the rest of the analytic structure unaltered.

All two character theories satisfy the Fuchs relation, (3.5), with $l = 0$. The indices of the one point function of the previous operator satisfy (3.5) which by repeated application gives $l = N$. For N=1 we are talking about the one-point function of a current. But there is no such equation with $l = 1$ as indicated above. Thus the one-point function of the currents must vanish. For $N = 2$ the Wronskian must have a single zero at the third order orbifold point. Thus the second order equation with this property is,

$$f'' + \left[\frac{2}{3} \frac{2x-1}{x(x-1)} + p_1 \frac{Q_i(x)}{x(x-1)Q_\rho(x)} \right] f' + p_2 \frac{Q_a(x)}{[x(x-1)Q_\rho(x)]^2} f = 0 \quad (3.10)$$

where p_1, p_2, a need to be determined. Fixing the indices at $x = 0$ gives $p_1 = -\frac{2}{3}$, $p_2 = -\frac{c(c+4)}{144}$. Since $l = 2$ the indices at $x = \rho$ must be $(0, 2)$ since the other possibility, $(1,1)$, contains logarithms. This determines a to be equal to ρ. Thus the equation is completely determined. The characters of the respective theory satisfy,

$$\chi'' + \frac{2}{3} \frac{2x-1}{x(x-1)} \chi' - \frac{c(c+4)}{144} \frac{Q_\rho(x)}{[x(x-1)]^2} \chi = 0 \quad (3.11)$$

Then it is not hard to show that if χ is a solution of (3.11) then $f = [x(x-1)]^{\frac{2}{3}} \frac{d\chi}{dx}$ is a solution of (3.10). The statement above transformed into the standard one-point function says that it is the derivative with respect to τ of the associated character. Since the stress-energy tensor is one of the descendants at level two, this is hardly surprising. What we showed though, is that any descendant at level two will have the same one-point function as the stress-tensor.

Let's continue and see what happens at $N = 3$. There the Wronskian must have a single zero at $\tau = i$. The index scheme there should be $(0,2)$.* Going through the same procedure as before we arrive at the equation,

$$f'' + \left[\frac{2}{3} \frac{2x-1}{x(x-1)} - \frac{Q_\rho^2(x)}{x(x-1)Q_i(x)} \right] f' - \frac{(c+6)(c-2)}{144} \frac{Q_\rho(x)}{[x(x-1)]^2} f = 0$$
$$(3.12)$$

However the solutions of (3.12) contain logarithmic singularities at $x = -1, 2, 1/2$ unless $c = 2$. In this case the two independent solutions are

---

*Two indices, in general, should never be equal. There is no way one can avoid logarithmic singularities in such a case.

$f_0 =$constant, $f_1 \sim Q_\rho(x)[x(x-1)]^{-\frac{2}{3}}$. Thus the only non-zero one-point function at level three over the identity exists for the $c = 2$ two-character theory. Of course this theory can be identified with the $k = 1$ $SU(3)$ critical WZW theory so the previous results is certainly not surprising. The important point is that it was derived without that knowledge. The differential equations "know" what kind of theory they are dealing with.

At $N = 4$ the Wronskian has a double zero at the third order point. There are two possible schemes, either (0,3) or (1,2). In the first case the equation is,

$$f'' + \left[\frac{2}{3}\frac{2x-1}{x(x-1)} - \frac{4}{3}\frac{Q_i(x)}{x(x-1)Q_\rho(x)}\right]f' - \frac{(c+8)(c-4)}{144}\frac{Q_\rho(x)}{[x(x-1)]^2}f = 0$$
(3.13)

This contains logarithmic singularities unless $c = 4$ (k=1, SU(8) WZW model), in which case, $f_0 =$constant, $f_1 = Q_i(x)[x(x-1)]^{-1}$. In the second case the equation is,

$$f'' + \left[\frac{2}{3}\frac{2x-1}{x(x-1)} - \frac{4}{3}\frac{Q_i(x)}{x(x-1)Q_\rho(x)}\right]f' - \frac{(c+8)(c-4)}{144}\frac{Q_a(x)}{[x(x-1)Q_\rho(x)]^2}f = 0$$
(3.14)

If we set $f(a) \equiv -\frac{a^6-3a^5+5a^3-3a+1}{a^2(a-1)^2}$ then $Q_a(x)$ in (3.14) is determined by $f(a) = 6\frac{c^2+4c-176}{(c+8)(c-4)}$. The solution to (3.14) is again simple in terms of the characters of the corresponding theory. That is,

$$f_{0,1}(x) = [x(x-1)]^{\frac{4}{3}}\left[\frac{d^2}{dx^2} + \frac{2}{3}\frac{2x-1}{x(x-1)}\frac{d}{dx}\right]\chi_{0,1}(x) \sim \frac{Q_\rho(x)}{[x(x-1)]^{\frac{2}{3}}}\chi_{0,1}(x)$$
(3.15)

For $N = 5$ the equation has apparent singularities at both orbifold points. Its explicit form is,

$$f'' + \left[\frac{2}{3}\frac{2x-1}{x(x-1)} - \frac{5}{3}\frac{Q_a(x)}{x(x-1)Q_i(x)Q_\rho(x)}\right]f' - \frac{(c+10)(c-6)}{144}\frac{Q_\rho(x)}{[x(x-1)]^2}f = 0$$
(3.16)

with a satisfying, $f(a) = \frac{33}{10}$. As before, when $c \neq 6$ the equation contains logarithmic singularities. Thus the one-point function at level five is only non-zero for the $c = 6$ theory in which case $f_0 =$constant, $f_1 = Q_\rho^2(x)[x(x-1)]^{-\frac{4}{3}}$.

We will finally discuss the $N = 6$ one-point functions since in this case the Wronskian, having $l = 6$ can a zero anywhere inside moduli space. We will distinguish three cases.

(i) The Wronskian has a triple zero at the third order point. There are two possible sets of indices at that point, (0,4) and (1,3). The (0,4) scheme has always logarithmic singularities. The equation with (1,3) is the following,

$$f'' + \left[\frac{2}{3} \frac{2x-1}{x(x-1)} - 2 \frac{Q_i(x)}{x(x-1)Q_\rho(x)} \right] f' + \frac{(8-c)(c+12)}{144} \frac{Q_a(x)}{[x(x-1)Q_\rho(x)]^2} f = 0$$
(3.17)

with $f(a) - 6 = 36^2/(8-c)(c+12)$. Its solutions are $f_{0,1} = Q_\rho(x)\frac{d}{dx}\chi_{0,1}$.

(ii) The Wronskian has a double zero at the second order point. There are two possible schemes at that point, (0,3) and (1,2). (0,3) turns out to always contain logarithmic singularities. The equation for the (1,2) scheme is,

$$f'' + \left[\frac{2}{3} \frac{2x-1}{x(x-1)} - 2 \frac{Q_\rho^2(x)}{x(x-1)Q_i(x)} \right] f' + \frac{(8-c)(c+12)}{144} \frac{Q_\rho(x)Q_a(x)}{[x(x-1)Q_i(x)]^2} f = 0$$
(3.18)

with $4f(a) + 3 = 7776/(8-c)(c+12)$. The solutions are $f_{0,1} \sim Q_i(x)[x(x-1)]^{-1}\chi_{0,1}$.

(iii) The Wronskian has a single zero anywhere else inside moduli space. Thus the associated differential equation will have an apparent singularity at a point a and its images. The scheme there can only be (0,2) and the equation with such a scheme and the appropriate indices at $x = 0$ is,

$$f'' + \left[\frac{2}{3} \frac{2x-1}{x(x-1)} - 2 \frac{Q_i(x)Q_\rho^2}{x(x-1)Q_a(x)} \right] f' - \frac{(c+12)(8-c)}{144} \frac{Q_\rho(x)Q_b(x)}{[x(x-1)]^2 Q_a(x)} f = 0$$
(3.19)

The extra undetermined parameter which is b in (3.19) is fixed by requiring absence of logarithmic singularities. It may seem surprising at first that the solution to (3.19) is in fact an appropriate linear combination of solutions of (3.17) and (3.18). Set $g(x) = Q_i(x)[x(x-1)]^{-1}\chi + \kappa Q_\rho(x)\frac{d}{dx}\chi$ where χ is a solution of (3.11). Then g is a solution of (3.19) with a determined implicitly as a function of κ and c,

$$f(a) = \frac{3(24\kappa^2\mu - 26\kappa - 3)}{4(3\kappa^2\mu - \kappa + 3)}, f(b) = \frac{3(72\kappa^2\mu^2 - 60\kappa^2\mu - 240\kappa\mu - 9\mu + 56\kappa + 156)}{4(3\mu + 2)(3\kappa^2\mu - \kappa + 3)}$$
(3.20)

This feature is an example of an isomonodromic deformation. All the equations (3.19) with a varying in moduli space generate the same representation

12

of the modular group which, as can be seen from the solution g is the same as the one of the respective characters. Later on we will use this property to mod-out this redundancy.

Finally let's study an example of a one-point function of a non-trivial primary operator and its descendants. Consider the primary operator of spin one in the $k = 3$ SU(2) WZW model. Due to the fusion rules, $\left[\frac{1}{2}\right] \otimes \left[\frac{1}{2}\right] = [0] \oplus [1]$, $[1] \otimes [1] = [0] \oplus [1]$ there are two blocks for the one-point function of any descendant of the spin-one operator. Let Φ_N be any secondary operator at level N. Then $\Delta_N = \Delta_1 + N = \frac{2}{5} + N$, the dimension of the spin one-half primary is $\Delta_{1/2} = \frac{3}{20}$ and the central charge is $c = \frac{9}{5}$. The leading behaviour as $q \to 0$ is as follows,

$$f_{1/2} \equiv \frac{\langle \Phi_N \rangle_{1/2}}{\eta^{2\Delta_N}} \sim q^{-\frac{2N-1}{24}}[1 + \mathcal{O}(q)] \qquad (3.21a)$$

$$f_1 \equiv \frac{\langle \Phi_N \rangle_1}{\eta^{2\Delta_N}} \sim q^{-\frac{2N-1}{24}+\frac{1}{4}}[1 + \mathcal{O}(q)] \qquad (3.21b)$$

The subscripts indicate the representation that goes around the loop. ¿From (3.5) we obtain that the Wronskian must have $\frac{N-1}{6}$ zeros in the interior of moduli space. Thus in the case of the primary itself the Wronskian must have a single pole somewhere. But this is not allowed, thus the one-point function must vanish. Again the reader might think that this is trivial since the primary transforms non-trivially under SU(2) so its one-point function was bound to vanish due to global SU(2) invariance. But as we argued in a previous situation what was used was a bunch of characters from which we could get the fusion rules à la Verlinde. Proceeding, at $N = 1$ we have a Wronskian with no zeros and the equation is nothing else but the equation for the characters of the $k = 1$ SU(2) WZW model! Thus the characters of the $k = 1$ theory are the one-point function blocks of the $k = 3$ theory. At $N = 2$, $l = 1$ and as before the one-point functions vanish. At $N = 3$, the one-point functions are covariant derivatives of the k=1 characters, $f_{1/2,1} \sim [x(x-1)]^{2/3}\frac{d}{dx}\chi_{0,1}$ and so on. One can be a little bit more general in the case of SU(2) WZW models. First any one-point function of an operator belonging to a half-integer spin representation is necessarily zero due to the fusion rules. For a descendant at level N of a representation with integer spin, I, the Wronskian must have $l = (k - 2I + 1)(N - I)/2$ zeros inside moduli space. It immediately follows that descendants up to level $I - 1$ have

13

vanishing one-point functions. The first non-zero one-point function occurs at level $N = I$.

At this point it is instructive to come back and understand the meaning of the bound on the number of apparent singularities. As mentioned above for a second order equation one can produce all possible monodromy representations by restricting to the case without any apparent singularities. To put it differently, if one constructs a second order differential equation with an arbitrary number of apparent singularities then its solutions are related to the solutions of a second order differential equation without apparent singularities by a combination of the two following procedures, taking (covariant) derivatives and/or multiplying by rational functions. Of course in our case we like to avoid poles in the interior of moduli space so we can multiply only by rational functions with poles at $0, 1, \infty$. This sheds also some light into the following question. If we are searching for equations whose solutions should serve as characters of some RCFT why not examine any possible equation instead of the minimum set that reproduces the relevant representations of the modular group? The answer is that if one found such an equation that its solutions have integral Fourier coefficients then either the solutions would be related to characters found already by multiplying by a polynomial in the j-function or they will be one-point functions of descendants of the identity in some other theory. It is, for example, obvious that the one-point function of the stress tensor of a RCFT will have blocks with integral Fourier coefficients. However such blocks cannot be used as characters. There are various reasons for this. One that they correspond to characters in a theory with a non-SL(2,C) invariant vacuum, second the fusion rules, N_{ijk} will fail to be non-negative integers.[†] The arguments above are not special to second order equations but hold in general, [18].

One can describe a general argument which does not rely on group invariance that one-point functions of dimension one chiral operators must vanish. If the number of blocks in a theory is n then using (3.5) one can show that the Wronskian of the equation for the one-point functions in question must have $l = \frac{n}{2}$. Since l must always be a non-negative integer, when n is odd the argument is obvious. If n is even, we assume that the zeros of the Wronskian are at the same point, that is, there is multiple zero there. If they are not

---

[†]This was verified explicitly in several cases of second order equations, [26], as for example the ones presented in [22].

then the following argument is even stronger. Let N_i, $i = 1, ..., n$ be the indices at the apparent singularity. They must be non-negative integers and not any two of them equal to each other (otherwise the solutions contain logarithms). The Wronskian has a zero at the apparent singularity of order at most $[n/4]^*$. This implies that $\sum_{i=1}^{n} N_i \leq (2n^2 - n)/4$. Suppose that none of the N_i is zero, then $\sum_{i=1}^{n} N_i \geq n(n+1)/2$ which is incompatible with the previous inequality. Thus one of the indices there must be zero. Using the general form of the differential equation, (3.2), we can see that the solutions will contain logarithms unless one of the indices around $x = 0$ is also zero. In that case the equation reduces to a $(n-1)$-order equation. The Wronskian though continues to have the same order of zeros inside moduli space. Since $n - 1$ now is odd, the statement is proven.

An important issue both with characters and one-point functions is the phenomenon encountered in the previous analysis of the one-point functions of descendants over the identity at level six. There, the differential equation had an apparent singularity at a point inside moduli space which could be continuously varied. This a special case of a so-called isomonodromic deformation. Since this is an important issue we are going to say a few more things about it.

In general consider a FDE that depends analytically on certain continuous parameters. The question of isomonodromic deformations is equivalent to finding the conditions under which the monodromy data are independent of the parameters mentioned above. A trivial case of the above is when there are apparent singularities that can be moved around. It is trivial in the sense that by arranging that a singularity be apparent one imposes that the monodromy there is trivial. However in general this will affect the monodromy around the non-apparent singular points. In practice what this means is the following. Consider a modular invariant differential equation that contains apparent singularities only at the third order orbifold point. Then there are continuous deformations of the solutions that have apparent singularities at generic points inside moduli space. An easy way to find such continuous families is the technique already employed. One solves the equations with (compatible) apparent singularities at the third and second order points and then constructs an interpolation.

There are less trivial cases of isomonodromic deformations. In such cases

---

*This happens if the apparent singularity is at the third order point.

there are real singularities that can be continuously varied. Such is the case for example for the equations for the two-point functions on the torus when written on the branched sphere as it is done in Appendix D. There there is a non-apparent singularity at a point that depends on the modulus of the torus. However since this modulus dependence was introduced via the transformation from the torus to the sphere and on the torus the singular point $z = 0$ is certainly fixed, this again corresponds to an isomonodromic deformation that is taken care of, automatically. Still it is amusing to note that this rather trivial case generates easily solutions to non-linear Painlevé type equations which are associated with isomonodromic deformations of second order FDE's.

Where the issue seems certainly to be non-trivial is in the case of higher correlation functions like five point functions on the sphere or genus two characters. We are going to discuss such cases in section 5.

4. TWO-POINT FUNCTIONS ON THE TORUS.

In this section we are going to analyze two-point functions of RCFT's on the torus. It is well known that a knowledge of two-point functions of the torus is enough to determine the structure constants of the CFT. Our aim will be to try to determine linear differential equations that the blocks of two-point functions satisfy. This approach was initiated in [16] where such questions were discussed. What we will show here is that the data coming from the characters are enough to determine the differential equations for the two-point functions and consequently the two-point functions themselves[†]. In the process we will explain what we need in order to write down such differential equations by giving a concrete algorithm.

To set up the problem we will consider the two-point function of an operator ϕ of dimension Δ on the torus. The two-point function can be non-zero in general even if the two operators are not the same. This turns out to provide no extra difficulties and we will comment on it in due time. Translational invariance on the torus being intact, the two-point function depends on the distance between the two insertion points, z, and the modulus of the torus, τ. The number of blocks can be easily calculated from the fusion rules. In the particular basis where we fuse first the operators ϕ, the blocks are labelled by

---

[†]This, in general does not mean that any data coming from the characters will give consistent two-point functions. There are extra constraints that they must satisfy, [8].

the representation coming out of the fusion process and the representation going around in the loop, (in a way consistent with the fusion rules), see fig. 1. Let $f_i(z)$, $i = 1, 2, ..., n$ be the blocks of the two-point function in question. They generate a representation of the mapping class group of the twice punctured torus. This group is generated by the action of the following transformations on the moduli space, $T_1 : z \to z + 1$, $T_2 : z \to z + \tau$ and the usual modular transformations, $S : \tau \to -\frac{1}{\tau}$, $z \to \frac{z}{\tau}$, $T : \tau \to \tau + 1$, $z \to z$.

The blocks can be viewed as the solution of an n-th order linear differential equation in the variable z,

$$\partial_z^n f + \sum_{i=1}^{n} q_i(z) \partial_z^{n-i} f = 0 \tag{4.1}$$

The following properties of (4.1) follow from the requirements of CFT[‡]. The coefficient functions are meromorphic functions on the torus, that is doubly-periodic (elliptic) functions with singularities which are poles. In fact (4.1) should be Fuchsian with the only regular singularity being at $z = 0$, any other singularities of the coefficient functions q_i being apparent singularities. In order for $z = 0$ to be a regular singular point $q_i(z)$ should behave around $z = 0$ as, $q_i(z) \sim z^{-i}$ or less singular. Analogous statements should be true around possible apparent singularities. (4.1) should also be invariant under the mapping class group in order that its solutions generate a representation of it. Invariance under $T_{1,2}$ is guaranteed by $q_i(z)$ being elliptic. Invariance under S, T implies that,

$$q_i(z, \tau + 1) = q_i(z, \tau) \tag{4.2a}$$

$$q_i(\frac{z}{\tau}, -\frac{1}{\tau}) = \tau^i q_i(z, \tau) \tag{4.2b}$$

¿From (4.2) we immediately learn that $q_i(-z) = (-1)^i q_i(z)$.

There are a few things on elliptic functions that will be of use[§]. Let $g(z)$ be an even elliptic function, that is $g(-z) = g(z)$. Then $g(z)$ can be uniquely determined by its zeros and poles elsewhere than $z = 0$. Its generic form is,

$$g(z) = \frac{\prod_{i=1}^{N}[\wp(z) - \wp(a_i)]}{\prod_{j=1}^{M}[\wp(z) - \wp(b_i)]} \tag{4.3}$$

[‡]For more details the reader is referred to [16].
[§]For more details see any book on elliptic functions

$\wp(z)$ is the standard Weierstrass function which is the unique elliptic function (up to a constant) which is holomorphic on the torus except at $z = 0$ where it has a double pole. If b_i lies on the a-homology cycle then $g(z)$ in (4.3) has a single pole at b_i and another single pole at $1 - b_i$. If b_i lies on the b-homology cycle then $g(z)$ has a single pole at b_i and another one at $\tau - b_i$. If b_i is a half-period, $(\frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2})$, then $g(z)$ has a double pole there. Finally if b_i is anywhere else on the torus then $g(z)$ has a single pole at b_i and another at $1 + \tau - b_i$. Similar remarks hold for the zeros of $g(z)$. The order of the pole of $g(z)$ at $z = 0$ is $2(M - N)$ in accord with the standard result that the number of zeros should be equal to the number of poles. If $g(z)$ is an odd elliptic function then it can be written as,

$$g(z) = \wp'(z) \frac{\prod_{i=1}^{M}[\wp(z) - \wp(a_i)]}{\prod_{j=1}^{N}[\wp(z) - \wp(b_i)]} \tag{4.4}$$

which has a pole of order $2(N-M)+3$ at $z = 0$. ¿From (4.4) it is obvious that there is no holomorphic function on the torus with a single pole at $z = 0$.[¶] The information above will be crucial in counting parameters in equation (4.1).

The coefficient functions $q_i(z)$ can be written in terms of the Wronskians, $W_i(z)$, as $q_i(z) = W_i/W$ where $W(z)$ is the standard Wronskian. In particular $q_1(z) = -\partial_z \log W$. $W_i(z)$ is an elliptic function with poles only at $z = 0$. As we will show in the sequel, we will be able to determine the order of the pole of $W(z)$ at $z = 0$ from knowledge of appropriate one-point functions. Suppose that $W(z)$ has a pole of order $2N$[§] at $z = 0$.[*] Our previous discussion of elliptic functions implies that, $W(z) \sim \prod_{i=1}^{N}[\wp(z) - \wp(z_i)]$. Then in order that (4.1) be Fuchsian the coefficient functions must be of the following form,

$$q_{2k}(z) \sim \frac{\prod_{i=1}^{N+k}[\wp(z) - \wp(w_i)]}{\prod_{j=1}^{N}[\wp(z) - \wp(z_j)]} \quad , \quad k \geq 1 \tag{4.5a}$$

$$q_{2k+1}(z) \sim \wp'(z) \frac{\prod_{i=1}^{N+k-1}[\wp(z) - \wp(w_i)]}{\prod_{j=1}^{N}[\wp(z) - \wp(z_j)]} \quad , \quad k \geq 1 \tag{4.5b}$$

[¶]This means that $z = 0$ is the Weierstrass point of the torus and the "gap" there is 1.
[§]Similar remarks hold for a pole of odd order.
[*]The Wronskian cannot have a zero at $z = 0$ because that would imply that it has a pole somewhere else, a situation that is not allowed in CFT.

18

while $q_1(z)$ is as always the logarithmic derivative of the Wronskian. The coefficients of proportionality in (4.5) should be z-independent modular invariants. In principle they should be rational functions of the j-function. But since we would like the indices of (4.1) to be ordinary numbers and not to vary with τ this fixes the coefficients to be just ordinary numbers. Thus simple counting shows that an equation of order $2n$ depends on $n^2 + (2N+1)n$ coefficients which consist of the zeros and poles of the coefficient functions $q_i(z)$ as well as the proportionality coefficients. If the order is $2n+1$ then the respective number of coefficients is, $n^2 + 2(N+1)n + N$. When the Wronskian has a pole of order $2N+3$ at $z=0$ then things change accordingly,

$$W(z) \sim \wp'(z) \prod_{i=1}^{N} [\wp(z) - \wp(z_i)] \tag{4.6}$$

while (4.5) are still true. The counting of parameters is also the same in this case.

The positions of the zeros parametrized by z_i of the Wronskian should be apparent singularities of (4.1). This implies some extra conditions on the parameters amounting to imposing that the monodromy matrix around them is trivial. For (4.1) the number of such conditions is $N(n-1)^{\|}$. Taking the above into account, an equation of order $2n$ depends on $n^2 + n + N$ undetermined parameters while one of order $2n+1$ on $n^2 + 2n + N$ parameters.

We will now come to the important issue of the indices of (4.1) around $z=0$. Let the fusion rule for our operator ϕ be, $[\phi] \otimes [\phi] = \oplus_i [\phi_i]$ where $[\phi_i]$ stand for representations of the chiral algebra of the theory. ¿From figure 1 we understand that for each family $[\phi_i]$ the number of blocks contributing will be equal to $\sum_j N_{ijj}$. In particular since the identity representation will always appear in the fusion process[**] and the identity couples to any representation going around in the loop, we will obtain at least m blocks for the two point function, where m henceforth will denote the number of representations in the theory. Let's focus now on the blocks generated by the intermediate family, $[\phi_i]$. We will write the contribution to the OPE $[\phi] \otimes [\phi]$ coming from the i-th family as,

$$\phi(z)\phi(0) \sim z^{-2\Delta+\Delta_i} \sum_n z^n \mathcal{O}_n(0) \tag{4.7}$$

[||]See for example, [24].

[**]This is because we study the two-point function of conjugate operators. If this is not the case, then the identity will not appear.

19

The previous equation is written schematically. OPE coefficients have been suppressed as well as sums at a given level of the representation. If we write the blocks as Laurent series in z then the coefficients will be appropriate one-point functions of the operators appearing in the fusion process. Let the number of blocks coming from the i-th family be N_i. Then their behaviour as $z \rightarrow 0$ will be of the form $z^{-2\Delta+\Delta_i+n_j}$, $j = 1, 2, ..., N_i$, and the n_j appearing will be the first N_i levels such that there exists at least one operator in that level with a non-zero one-point function. The integers n_j have been called "gaps" in [16] and have been assumed to be additional input information. As our discussion in the previous section made clear these numbers are determined from the character data.

A rather simple example might help the reader understand the basis for the previous discussion[††]. Consider the $k = 1$ SU(2) WZW model. Let ϕ be the spin one-half primary field. Then, since $[\frac{1}{2}] \otimes [\frac{1}{2}] = [0]$ there are two blocks in its two-point function corresponding to the identity coupling to the two representations going around in the loop. In (4.7) the first contribution comes from the identity itself giving a behaviour $z^{-1/2}$. The next contribution would come from a level-one operator in the identity representation, that is a current. But one-point functions of currents vanish. Thus we must look at level-two where we get the stress-tensor with a non-vanishing one-point function. Consequently the other block behaves as $z^{-\frac{1}{2}+2}$.

Thus the procedure described in the previous section is useful in obtaining the indices at $z = 0$ of (4.1). Their knowledge has a two-fold purpose. First it determines the order of the pole of the Wronskian at $z = 0$ and hence the number N defined above, (see (4.5)). Second, it determines the multiplicative constants in (4.5) by the requirement that the indicial equation at $z = 0$ has as solutions the by now known indices. ¿From now on, we will assume for concreteness that the order of the FDE is $2n$. The indices consequently fix $2n - 1$ parameters in the equation and we are left with $n^2 - n + 1 + N$ parameters. In order to restrict more our equation we will have to study its degeneration when the two-punctured torus degenerates to the four-punctured sphere.

The limit to consider is when $q = e^{2\pi i\tau} \rightarrow 0$ which corresponds to pinching the torus along the a-cycle. We will need the q-expansion of the Weierstrass

---

[††]Many more examples can be found in [16].

20

function,

$$\wp(z) = \pi^2 \left[\frac{1}{\sin^2 \pi z} - \frac{1}{3} - \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}(\cos(2n\pi z) - 1) \right] \qquad (4.8)$$

The proper coordinate on the four-punctured sphere is $x = e^{2\pi iz}$. Consider a specific block of the two-point function corresponding to the two operators fusing into the i-th representation which then couples to the j-th representation going around the loop as shown in fig. 1. We will denote it, in short-hand, as $f_{ij}(z)$. When the surface degenerates we obtain,

$$f_{ij}(z) \sim q^{-\frac{c}{24}+\Delta_j} \left[x^\Delta \langle j|\phi(1)\phi(x)|j\rangle_i + \mathcal{O}(q) \right] \qquad (4.9)$$

which follows from the precise definition of the block in the Hamiltonian formalism, [7]. The matrix element in (4.9) is the standard matrix element on the sphere with in and out states normalized a là BPZ. The subscript i indicates the i-th channel in the four-point correlation function. Thus, upon degeneration, the total of two-point blocks, f_{ij}, splits into collections of four-point blocks, the number of collections being m, the number of representations in the theory. In particular the $f_{00}(z)$ block degenerates into the two-point function of ϕ on the sphere,

$$f_{00}(z) \sim q^{-\frac{c}{24}} \left[x^\Delta (1-x)^{-2\Delta} + \mathcal{O}(q) \right] \qquad (4.10)$$

Using,

$$x \equiv e^{2\pi iz} \ , \ \ \partial_z^n = (2\pi i)^n \left(x\frac{\partial}{\partial x} \right)^n + \mathcal{O}(q) \qquad (4.11a)$$

$$\wp(z) = -\frac{\pi^2}{3}\frac{x^2 + 10x + 1}{(x-1)^2} + \mathcal{O}(q) \ , \ \ \wp'(z) = -(2\pi i)^3 \frac{x(x+1)}{(x-1)^3} + \mathcal{O}(q) \quad (4.11b)$$

we obtain the equation on the sphere which is satisfied by the appropriate four-point functions.

As we mentioned before for every representation j in the theory we obtain a number $N_j \equiv \sum_i N_{\phi\phi i}N_{ijj}$ of four-point blocks corresponding to the four-point function, $\langle i|\phi\phi|i\rangle$[¶]. Obviously, $N_0 = 1$, corresponding to the two point function and $2n = 1 + \sum_{j=1}^{m-1} N_j$. Now if we impose that the equation on

---

[¶]When the ground state of the representation $[i]$ is degenerate then what we obtain is a some over all components.

the sphere, obtained as the $q = 0$ limit of (4.1) has as solution the two-point function this implies $2n + N$ extra conditions on the so far undetermined coefficients. Thus we are left with an $2n - 1$ order equation and $n^2 - 3n + 1$ undetermined as yet coefficients. It is obvious that up to a fourth order equation the data so far would have been enough*. In Appendix A we work out explicitly two examples of degenerations of (4.1) for the $k = 1$ SU(3) WZW model and the $k = 1$ G_2 WZW model in order to give a flavour of techniques that come into play.

In the sequel we will show that factorization of the $2n - 1$ order equation into $m - 1$ equations each one describing the blocks of each one of the four-point functions is enough to determine the rest of the coefficients. In order to do this we will have to remind the reader of a few simple counting techniques in FDE's†.

Consider a FDE of order n on the sphere with three regular points which can be placed at $0, 1, \infty$. There are n indices at each singular point, in total $3n$ but they satisfy the Fuchs' relation so that only $3n - 1$ are independent. A simple counting of coefficients in such an equation reveals that their number is $E = n(n + 3)/2$. On the other hand an arbitrary monodromy representation ρ which is a homeomorphism from the fundamental group of the three-punctured sphere into $GL(n, \mathbf{C})$ depends on $M = n^2 + 1$ parameters. The reason is the following. There are two independent generators of the fundamental group which could be taken to be the loops around $x = 0, 1$. To each one of them we associate a matrix. Thus we have so far $2n^2$ complex coefficients. But these matrices are defined up to overall conjugation‡ which can be used to get rid of $n^2 - 1$ coefficients. The difference $M - E \geq 0$ is taken care of by apparent singularities. Consider an equation of order n. The question we would like to answer is, how many conditions we must impose among its coefficients so that it factorizes into two equations of orders n_1, n_2 with $n = n_1 + n_2$? A precise definition of "factorization" is equivalent to the statement that the monodromy representation is completely reducible. The monodromy representation of the initial equation depends on $n^2 + 1$ pa-

---

*In [16] equations up to fourth order have been considered. Thus it was enough to impose the constrains mentioned so far. At order five one will have undetermined coefficients left. This did not appear in the single example of a fifth order equation discussed in [16] since one of the indices was zero in that case.

†More details can be found in [24].

‡This corresponds to the freedom of changing basis.

rameters. The representations of the two equations into which it factorizes depend on $n_1^2 + n_2^2 + 2$ parameters. Thus in order that the equation factorizes we must impose $2n_1 n_2 - 1$ conditions. The statement easily generalizes when the equation factorizes to more than two pieces.

We will now come back to our counting of parameters in (4.1). After we fixed the indices at $z = 0$, and imposed that the two point function be a solution we are left with a $(2n - 1)$-th order equation on the sphere describing the blocks of the $m - 1$ four-point functions and depending on $n^2 - 3n + 1$ parameters. The indices of this equation at $x = 0$ are already fixed and equal to the indices of the original equation at $z = 0$. However it still remains to fix the indices around $x = 1, \infty$. For each of the four-point functions, $\langle i | \phi \phi | i \rangle$, there are $2N_i - 1$ indices to be fixed, (N_i is the number of blocks of the four-point function). However (4.1) is invariant under $z \to -z$. ¿From (4.11a) this implies that the equation on the sphere obtained by degeneration is invariant under $x \to 1/x$ so that the indices around $x = 0$ are the same as those around $x = \infty$. Thus the independent indices that have to be fixed are $N_i - 1$ for each of the four-point functions. In total we obtain $\sum_{i=1}^{m-1}(N_i - 1) = 2n - m$ conditions and thus we are left with $n^2 - 5n + m + 1$ parameters. The last and crucial condition is that the equation on the sphere should factorize to the appropriate four-point functions as $2n - 1 = \sum_{i=1}^{m-1} N_i$. According to our previous discussion this implies another $2\sum_{i<j}^{m-1} N_i N_j - m + 2$ conditions. Thus the final number of undetermined parameters left is $R = n^2 - 5n - 2\sum_{i<j}^{m-1} N_i N_j + 2m - 1$. We will now prove that $R \le 0$ and thus the system is generically over-determined.[‡] There is a constraint on the magnitudes of N_i, they must be no-larger than the number of representations, m, present in the theory. The maximal value of R is obtained by minimizing $2\sum_{i<j} N_i N_j - 2m$ under the constraints, $\sum_i N_i = 2n - 1$, $1 \le N_i \le m$. In appendix B we prove that the required minimum is equal to $3n(n - 3)$ giving,

$$R \le -(2n^2 - 4n + 1) \le 0 \tag{4.12}$$

The same arguments and result apply when the order of (4.1) is odd. We have thus showed that the data from the characters are enough to determine the differential equation for the two-point function of an operator on the torus.

[‡]This may be another potential source of consistency conditions.

An issue which we have not discussed so far is the τ dependent normalization of the blocks. This can be immediately determined from the knowledge of the appropriate one-point functions and characters. For example,

$$f_{0i}(z) \sim z^{-2\Delta+n_i} \langle \Phi_{n_i} \rangle [1 + \mathcal{O}(z)] \tag{4.13}$$

where Φ_{n_i} is a descendant over the identity at level n_i.

Of course there can be non-zero two-point functions on the torus where the two operators are not conjugate. In this case the counting is slightly different but as the reader can verify the same results apply. In Appendix C we work explicitly an example to illustrate the case.

The analysis above demonstrates our ability to compute the basic data that define the RCFT, (structure constants, fusion and braiding matrices etc.), from the basic data coming from the characters. In the next section we are going to discuss other correlation functions.

5. DIFFERENTIAL EQUATIONS FOR OTHER CORRELATION FUNCTIONS

¿From our experience with CFT, we know that once we know four-point functions on the sphere and one-point functions on the torus we can, in principle, compute any other correlation function, [11]. In practice things may present difficulties. Correlation functions that depend on more than one variable are certainly more difficult to calculate. The case considered in the previous section concerning the two-point functions on the torus presented some simplified features which made our life rather easy.

There are two possible forms of linear differential equations that one may attempt to construct for correlation functions depending on more than one variable. One form could be a linear partial differential equation. However this form is rather difficult to derive, especially since the complete solution to the R-H problem in that case is unknown.* A more promising approach seems to be deriving an ordinary differential equation in one of the variables and treating the dependence through the rest of the variables using the theory of isomonodromic deformations.

---

*In cases when one knows the explicit form of the null vectors of the chiral algebra then one can construct such a partial differential equation, [27,28]. However such knowledge is not always at hand.

Consider a correlation function depending on the variables x, t_i, $i = 1, 2, \cdots, n$. The correlation function viewed as a function of x will satisfy a FDE with singularities and coefficients depending on the t_i. Despite the fact that the parameters t_i are free to move around in a continuous manner the monodromy of the correlation function is independent of them. This poses rather severe constraints on the dependence of the parameters of the equation on t_i. To be more concrete consider the second order equation,

$$\partial_x^2 f + p_1(x, t)\partial_x f + p_2(x, t)f = 0 \qquad (5.1)$$

which depends on a complex parameter t in such a way that the monodromy is independent of t. An intuitive way of stating this condition is that if we change t infinitesimally the variation of the solutions continue to have the same monodromy. Consequently the variation must be a linear combination of the initial solution and its derivatives with coefficients being rational functions in x and analytic in t. Since in the case at hand the only independent derivative is the first we can write the condition of an isomonodromic deformation as, [29],

$$\partial_t f = A(x, t)\partial_x f + B(x, t)f \qquad (5.2)$$

where A, B are rational in x and analytic in t. The existence of A, B implies that the system,

$$\partial_x f = y \ , \ \partial_x y = -p_1 y - p_2 f \qquad (5.3)$$

and (5.2) is completely integrable, a statement that translates after straightforward algebra to,

$$\partial_x^2 B + p_1\partial_x B - 2p_2\partial_x A - (\partial_x p_2)A = -\partial_t p_2 \qquad (5.4a)$$

$$2\partial_x B + \partial_x^2 A - p_1\partial_x A - (\partial_x p_1)A = -\partial_t p_1 \qquad (5.4b)$$

If we eliminate B from (5.4) we obtain a linear non-homogeneous equation for A,

$$\frac{1}{2}\partial_x^3 A - 2p\partial_x A - (\partial_x p)A = -\partial_t p \qquad (5.5)$$

with,

$$p(x, t) = \frac{1}{2}\partial_x p_1 + \frac{1}{4}p_1^2 - p_2 \qquad (5.6)$$

25

which can be solved to obtain a rational solution. Then B can be found from (5.4). Notice that B can be determined up to an additive function of t corresponding to the freedom to normalize the function f by an arbitrary function of t. When the order of the equation is greater than two one has to include in the right-hand side of (5.2) a linear combination of the linearly independent derivatives. If the parameters are more than one then (5.2) is replaced by a matrix equation. However all the properties and procedures discussed above go through unaltered, [30].

An example will give a feeling of the situation. Consider the following five-point function in the Ising model, $\langle \sigma(x)\sigma(y)\sigma(1)\sigma(0)\psi(\infty)\rangle^\dagger$. ¿From the fusion rules we learn that there are two blocks in this correlation function. Viewed as a function of x it has singularities at $0, 1, \infty, y$. The most general FDE with the singularities as above is,

$$\partial_x^2 f + \left[\frac{a_1}{x} + \frac{a_2}{x-1} + \frac{a_3}{x-y}\right]\partial_x f + \frac{C_1 x^4 + C_2 x^3 + C_3 x^2 + C_4 x + C_5}{[x(x-1)(x-y)]^2}f = 0 \tag{5.7}$$

The coefficients C_i depend on y. The dependence of (5.6) on y must be analytic. Thus the C_i are polynomials in y. The indices at $0, 1, y$ can be easily found to be $-1/8, 3/8$ whereas the indices at ∞ are $1/8, 9/8$ This fixes all the functions except C_2,

$$a_1 = a_2 = a_3 = \frac{3}{4} \tag{5.8a}$$

$$C_1(y) = \frac{9}{64} \; , \; C_3(y) = -\frac{3}{32}(2y^2 + y + 2) - (y+1)C_2(y) \tag{5.8b}$$

$$C_4(y) = \frac{3}{16}y(y+1) + yC_2(y) \; , \; C_5(y) = -\frac{3}{64}y^2 \tag{5.8c}$$

Analyticity of the equation as $y \to \infty$ implies that $C_2(y)$ can be at most linear in y, $C_2(y) = \kappa_1 y + \kappa_0$. Thus we are left with two undetermined constants. These are fixed by solving the isomonodromic equation, (5.5). We obtain,

$$\kappa_1 = \kappa_0 = -\frac{3}{16} \; , \; A(x,y) = -\frac{x(x-1)}{y(y-1)} \; , \; B(x,y) = -\frac{x}{8y(y-1)} + g(y) \tag{5.9}$$

---

†We are using notation and conventions of BPZ.

so that the complete equation reads,

$$\partial_x^2 f + \frac{3}{4}\left[\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-y}\right]\partial_x f + \frac{3}{64}\frac{3x^4 - 4(y+1)x^3 + 6yx^2 - y^2}{[x(x-1)(x-y)]^2}f = 0$$
(5.10)

In the case above, the reader might wonder why do we go through all this isomonodromic machinery since we could have fixed the remaining coefficients by looking at the limits $y \to 0, 1, \infty$? The answer is that this might have been possible here but not in cases where the number of blocks exceeds two. The reason is that from each term in the relevant equation, four of the coefficient functions are fixed by the indices. However the rest of the coefficients are polynomials in y of degree generically greater than two, while the limiting procedure $y \to 0, 1, \infty$ gives only information about the values of the relevant polynomials at these three points which in general is not enough to determine them.

Isomonodromic relations of the form (5.2) are crucial in determining contour-integral representations for the solutions of equations like (5.10). The correct y-dependent normalization can again be computed from the factorization of the five-point function over a four-point function.

The techniques employed above for 5-point functions have general validity, [30] and we assume that the reader can certainly see how to deal with higher point functions both on the sphere and at higher genus.

6. CONCLUDING REMARKS

In this work we tried to systematize the procedure for obtaining an "exact solution" of any RCFT. The "exact solution" is meant as our ability to write for any specific correlation function a linear Fuchsian differential equation whose solution proceeds along standard lines. The existence of such an equation is guaranteed by what we mean by RCFT as argued in section 2. In particular we tried to determine differential equations for correlation functions from what seems by now to be a minimal amount of data, that is the knowledge of the characters on the torus. In fact all we really needed was the central charge, the dimensions of the primary fields and the fusion rules which can be determined from the modular transformation matrices of the characters, [6].

The correlation function of central interest is the two-point function on torus (as well as its four-point function off-springs on the sphere). In order

27

to obtain enough information about it we were led to develop a very useful scheme for analysing one-point functions on the torus. The approach is a direct application of the formalism developed for the characters, [16,18], along with some new features coming in like the necessity of apparent singularities and the issue of the nature of continuously varying apparent singularities. In particular we demonstrated how to learn about the vanishing or not of specific one-point functions that are important in order to determine the local data necessary for two-point functions on the torus.

The knowledge of the indices of the two-point function is not enough to determine the appropriate FDE. We had to look more closely to the behaviour of the equation under the degeneration of the torus to the four-punctured sphere. One obtains this way a collection of four-point functions. The conditions that the resultant equation on the sphere have monodromy that is appropriately reducible provides us with enough constraints to fix the form of the equation completely. In that case one can evaluate in a straightforward manner the structure constants of the theory. On the side, we gave a procedure to solve the equations for the two-point functions by mapping them on the branched sphere where one can use standard contour-integral methods.

We investigated the problem of determining higher-point functions and gave an algorithm in order to derive the appropriate differential equations. It turns out that the theory of isomonodromic deformations is a useful tool in order to deal with correlation functions that depend on more than one variables.

There are some more refinements of the present work that might be interesting to consider. ¿From the work of Verlinde we know that the modular transformation matrices representing the transformation properties of the characters on the torus are algebraic over the integers. However even if a differential equation for characters has rational indices and coefficients, generically the monodromy matrices turn out to be transcendental. There exist some partial results on the subject pointing to the fact that algebraic monodromy representations arise when the differential equation does not depend on accessory parameters. To state it more simply, the central charge and the dimensions of the primary fields should determine the equation completely. In particular the modular transformation matrices (and thus the fusion rules) will be derivable from the knowledge of the critical dimensions and the central charge. This would certainly make the classification problem

potentially tractable. When the monodromy group is algebraic then there always exists a contour -integral representation of the correlation functions where the integral kernels are rational functions.

Another direction, as far as the classification of RCFT's is concerned, is to analyze polynomial equations satisfied by characters. The coefficients of such equations belong to the ring of modular functions that are holomorphic inside moduli space. The characters of G/H models certainly satisfy such polynomial equations since they transform trivially under a subgroup of finite index of the modular group. It is plausible that this property is shared by all RCFT's[‡]. In that case one can use Galois theory to transfer the classification problem of such polynomial equations to a group theoretic problem concerning the classification of automorphism groups of field extensions defined by the polynomial equation. Thus the problem can be reduced this way to the theory of finite groups. Some remarks pointing in this direction were also made recently by Atiyah in a slightly different context.

A very interesting question is going beyond RCFT's. The first step would be to consider quasi-rational CFT's, [8]. In such theories the same tools as above could be used in the analysis of correlation functions on the sphere. However, novel techniques seem to be required in dealing with higher genus objects. There is at least in principle a potential way of trying to construct non-rational CFT's with a given maximal chiral algebra. It is an extension of a nice idea by Zamolodchikov. In [31] he constructed a recursion formula for four-point blocks of the Virasoro algebra by using analyticity properties and the knowledge of the zeros of the Kač determinant. This can easily be extended to arbitrary chiral algebras by splitting the problem into two parts. First find the zeros of the appropriate Kač determinant and deriving a recursion formula. Second imposing monodromy invariance. The latter part is certainly non-trivial to implement at the moment.

Finally the issue of the relation between RCFT's and integrable models obtained as deformations of the former seems to a be quite interesting. We think that some of the issues discussed in this work, (especially isomonodromic deformations), seem to be promising in order to understand the aforementioned connection.

Acknowledgement

Appendix A

In this appendix we are going to work out explicitly two specific examples in which the degeneration of equation (4.1) will be studied, in order to make the relevant issues hopefully transparent.

We will first consider the $k = 1$ SU(3) WZW model and in particular the two-point function of the 3 and $\bar{3}$ primary fields of dimension 1/3. Since $[3] \otimes [\bar{3}] = [0]$, only the identity appears as an intermediate state and couples to all the representations going around the loop. Thus there are three blocks. The leading behaviour as $z \to 0$ is obtained when the identity operator appears in $[3] \otimes [\bar{3}]$ and is $\sim z^{-\frac{2}{3}}$. Since a current has a vanishing one-point function the next contribution comes from the stress tensor at level two, which gives, $\sim z^{-\frac{2}{3}+2}$. As we also saw in the previous section the model has a non-vanishing one-point function over the identity at level three giving $\sim z^{-\frac{2}{3}+3}$. Thus the Wronskian must be holomorphic on the torus and hence a constant. Consequently we obtain from our previous analysis,

$$q_1(z) = 0 \ , \ q_2(z) = a_1[\wp(z) - \wp(z_o)] \ , \ q_3(z) = a_2\wp'(z) \qquad (A.1)$$

and by fixing the indices at zero, $a_1 = -4/3$, $a_2 = -28/27$. Thus the equation is,

$$\partial_z^3 f - \frac{4}{3}[\wp(z) - \wp(z_o)]\partial_z f - \frac{28}{27}\wp'(z)f = 0 \qquad (A.2)$$

We should remember that z_o depends on τ. Using the degeneration formulae, (4.11), (4.13) can be cast in the form,

$$f''' + \frac{3}{x}f'' + \left[\frac{1}{x^2} + \frac{4}{3}\frac{x_o(x - x_o)(x - 1/x_o)}{(x_o - 1)^2 x^2(x - 1)^2}\right] f' + \frac{28}{27}\frac{x + 1}{x^2(x - 1)^3}f = 0 \ (A.3)$$

where $e^{2\pi i z_o(\tau)} = x_o + \mathcal{O}(q)$. Defining $f(x) = x^{1/3}g(x)$, by (4.9) $g(x)$ can be identified as the four-point matrix elements. $g(x)$ satisfies,

$$g''' + \frac{4}{x}g'' + \frac{7}{3x^2}g' + \frac{4}{3}\frac{x_o(x - x_o)(x - 1/x_o)}{(x_o - 1)^2 x^2(x - 1)^2}\left(g' + \frac{g}{3x}\right) + \frac{x^3 + 25x^2 + 31x - 1}{27x^3(x - 1)^3}g = 0$$
$$(A.4)$$

Imposing that the two point function $g(x) = (x-1)^{-2/3}$ be a solution implies that $x_o^2 + 10x_o + 1 = 0$ or $\wp(z_o) = 0$, in agreement with [16].

Now in order to factorize (A.4) explicitly we set $g(x) = (x-1)^{-2/3}h(x)$. Then (A.4) becomes,

$$h''' + \frac{2(x-2)}{x(x-1)}h'' + \frac{2}{9}\frac{x^2 - 2x + 10}{x^2(x-1)^2}h' = 0 \qquad (A.5)$$

which as expected has one solution $h(x) =$constant. It is easy now to find a second order equation whose solutions satisfy (A.5). The general procedure is to write a general second order equation, differentiate once using the equation again to get rid of the term with no derivative and then matching coefficients. This gives in our case,

$$h'' - \frac{2}{x(x-1)}h' + \frac{2}{9}\frac{1}{x^2}h = 0 \qquad (A.6)$$

which is solved by,

$$h_1(x) \sim x^{-\frac{1}{3}}(x+2) \ , \ \ h_2(x) \sim x^{-\frac{2}{3}}(2x+1) \qquad (A.7)$$

Since the $k = 1$ $SU(3)$ WZW model is effectively a free theory we are going to discuss a second example of a theory which is not free[**]. The model is the $k = 1$ G_2 WZW model. This is a two character theory whose characters have been analyzed in [16,18,22]. As before we are not going to assume any other knowledge except the central charge, $c = 14/5$, the dimension of the single non-trivial primary field, $\Delta = 2/5$ and the fusion rules, $[2/5] \otimes [2/5] = [0] \oplus [2/5]$, all of which can be derived from the equation from the characters. We will consider the two-point function of the $\Delta = 2/5$ primary field. According to the fusion rules there are three blocks for this correlator that are shown in fig. 2. In order to determine the Wronskian of the third order equation we need to know the first descendant of the identity representation as well as the first member of the non-trivial representation that have a non-vanishing one point function on the torus. In Section 3 we showed that the first non-zero one-point function over the identity occurs at level two. Thus the indices at $z = 0$ for the first two blocks in fig. are $-4/5$ and $2 - 4/5$. Consider now a descendant of the $2/5$ field at level N. Its one-point function has a single block. Another application of (3.5) shows that the number of zeros of the one-point function inside moduli space is

---

[**]That is, it cannot be represented in terms of free fermions or bosons and their orbifolds.

$l = (N - 3)/2$. Since $l \geq 0$ the first non-zero one point function occurs at level $N = 3$ and is given by $\eta^{4/5}$. Thus the index at $z = 0$ of the third block is $3 - 2/5$. The Wronskian has a pole at zero of order $-\frac{4}{5} + 2 - \frac{4}{5} + 3 - \frac{2}{5} - 3 = 0$ so it must be constant. As before the equation is,

$$\partial_z^3 f - \frac{48}{25}[\wp(z) - \wp(z_o)]\partial_z f - \frac{156}{125}\wp'(z)f = 0 \qquad (A.8)$$

Upon degeneration (A.8) obtains a form analogous to (A.3). The two point function $x^{2/5}(x-1)^{-4/5}$ must be a solution and this fixes $\wp(z_o) = 0$. Defining $f(x) = x^{2/5}(x-1)^{-4/5}g(x)$ we obtain for g,

$$g''' + \frac{3}{5}\frac{3x - 7}{x(x - 1)}g'' + \frac{3}{25}\frac{x^2 - 2x + 21}{[x(x-1)]^2}g' = 0 \qquad (A.9)$$

(A.9) implies that $g(x)$ satisfies a second order equation which is easy to determine,

$$g'' - \frac{1}{5}\frac{x + 11}{x(x - 1)}g' + \frac{8}{25}\frac{1}{x^2}g = 0 \qquad (A.10)$$

(A.10) is solvable in terms of hypergeometric functions. Thus the three solutions of the original equation are,

$$g_0(x) \sim x^{\frac{2}{5}}(x - 1)^{-\frac{4}{5}} \ , \ g_1(x) \sim (x - 1)^{-\frac{4}{5}}F(-6/5, -4/5, 7/5, x) \qquad (A.11a)$$

$$g_2(x) \sim x^{-\frac{2}{5}}(x - 1)^{-\frac{4}{5}}F(-6/5, -8/5, 3/5, x) \qquad (A.11b)$$

Appendix B

In this appendix we will discuss the minimization of the quantity $F_n \equiv 2\sum_{i<j}^{m-1} N_i N_j - 2m$ subject to the constraints, $\sum_i^{m-1} N_i = 2n - 1$ and $1 \leq N_i \leq m$ where N_i, m, n are positive integers and n fixed.

First we will show that for m fixed, F_n is minimized when $m - 2$ of the N_i are one and the remaining one $2n - m + 1$ so that the total sum is satisfying the constraint. In order to do that we substitute $N_1 = 2n - 1 - \sum_{i=2}^{m-1} N_i$,

$$F_n = 2(2n - 1)\sum_{i=2}^{m-1} N_i - \sum_{2=i<j}^{m-1} (N_i - N_j)^2 - 2m \qquad (B.1)$$

As it is obvious from (B.1) the minimum occurs when all except one of the differences $N_i - N_j$ are zero, substantiating our previous claim. What remains to be done is to scan for the minimum in m. This is quite easy, since for the optimal choice of N_i, $F_n = 4n(m - 3) + m - m^2$ whose minimum (taking into account that $1 \leq N_i \leq m$) is obtained when $m = n + 1$ giving $F_n^{min} = 3n(n - 3)$.

Appendix C

In this Appendix we are going to discuss in detail an example of a two-point function on the torus of two operators that are not conjugate to each other. We will consider the tricritical Ising model which is the second model of the minimal model series with $c < 1$. This model has $c = 7/10$ and the spectrum consists of primary fields with dimensions, $\frac{3}{80}, \frac{7}{16}, \frac{1}{10}, \frac{3}{5}, \frac{3}{2}$ along with the identity. The following fusion rules are of interest,

$$[\frac{7}{16}] \otimes [\frac{7}{16}] = [0] \oplus [\frac{3}{2}] \ , \ [\frac{3}{80}] \otimes [\frac{3}{80}] = [0] \oplus [\frac{1}{10}] \oplus [\frac{3}{5}] \oplus [\frac{3}{2}] \qquad (C.1a)$$

$$[\frac{3}{2}] \otimes [\frac{3}{2}] = [0] \ , \ [\frac{1}{10}] \otimes [\frac{1}{10}] = [0] \oplus [\frac{3}{5}] \qquad (C.1b)$$

$$[\frac{3}{5}] \otimes [\frac{3}{5}] = [0] \oplus [\frac{3}{5}] \ , \ [\frac{1}{10}] \otimes [\frac{3}{5}] = [\frac{1}{10}] \oplus [\frac{3}{2}] \qquad (C.1c)$$

The two-point function of a 1/10 and a 3/5 operator has according to (C.1) three blocks which are shown in fig. 3. In order to study the pole structure of the Wronskian we will first need to study one-point functions of the $[\frac{1}{10}]$ and $[\frac{3}{2}]$ families on the torus. For the $[\frac{1}{10}]$ family we will need to know the member at the smallest level that has a non-zero one-point function. In this case the one-point function has one block corresponding to the $[\frac{3}{80}]$ family going around the loop as it can be verified from (C.1) so it satisfies a first order equation. Using (3.5) we find that $l = level/2$, thus the primary has a non-zero one point function which is proportional to $\eta^{\frac{1}{5}}$.

For the $[\frac{3}{2}]$ family we need to know the first two non-vanishing one-point functions. There are two blocks here corresponding to the families $[\frac{3}{80}]$ and $[\frac{7}{16}]$ going around the loop. Consider a descendant at level N. Another application of (3.5) gives $l = N$. Thus the primary itself has a non-zero one-point function which satisfies a second order equation and its Wronskian has no zeros inside moduli space. The indices are $-\frac{7}{30}, \frac{17}{30}$. This fixes the equation to be,

$$f'' + \frac{2}{3} \frac{2x - 1}{x(x - 1)} f' - \frac{119}{900} \frac{Q_\rho(x)}{[x(x - 1)]^2} f = 0 \qquad (C.2)$$

its solution being written in terms of the standard hypergeometric functions,

$$f_1(x) \sim [x(x - 1)]^{-\frac{7}{30}} F(-7/10, 1/10, 1/5, x) \qquad (C.3a)$$

$$f_2(x) \sim [x(x-1)]^{\frac{17}{30}} F(9/10, 17/10, 9/5, x) \qquad (C.3b)$$

The next non-zero one-point point function exists at level $N = 2$, its Wronskian having a single zero at the third order point. It satisfies the following equation,

$$g'' + \frac{2}{3}\left[\frac{2x-1}{x(x-1)} - \frac{Q_i(x)}{x(x-1)Q_\rho(x)}\right]g' - \frac{119}{900}\frac{Q_\rho(x)}{[x(x-1)]^2}g = 0 \qquad (C.4)$$

The solutions of (c.4) can be written in terms of the solutions of (C.3),

$$g_{1,2}(x) \sim [x(x-1)]^{\frac{2}{3}}\frac{\partial}{\partial x}f_{1,2}(x) \qquad (C.6)$$

Armed with the information above we can find the order of the pole of the Wronskian at $z = 0$. The $[\frac{1}{10}]$ block behaves as $z^{-\frac{3}{5}}$ while the other two blocks as $z^{\frac{4}{5}}$, $z^{\frac{4}{5}+2}$. Thus the Wronskian has no pole at $z = 0$ and it must be a constant. The third order equation with the correct indices at $z = 0$ and its Wronskian being constant is,

$$\partial_z^3 f - \frac{48}{25}[\wp(z) - \wp(z_o)]\partial_z f - \frac{84}{125}\wp'(z)f = 0 \qquad (C.7)$$

Here we have so far one more undetermined parameter. It is again necessary to study the degeneration limit of (C.7). Going through the same steps as in Appendix A we arrive at an equation on the sphere which reads,

$$f''' + \frac{3}{x}f'' + \left[\frac{1}{x^2} + \frac{48}{25}\frac{x_o(x-x_o)(x-1/x_o)}{(x_o-1)^2x^2(x-1)^2}\right]f' + \frac{84}{125}\frac{x+1}{x^2(x-1)^3}f = 0 \quad (C.8)$$

where x_o is defined the same way as in Appendix A. The three solutions to (C.8) are identified with the single-block four-point function, $x^{3/5}\langle\frac{7}{16}|\phi_{3/5}(x)\phi_{1/10}(1)|\frac{7}{16}\rangle$ and the two-block four point function, $x^{3/5}\langle\frac{3}{80}|\phi_{3/5}(x)\phi_{1/10}(1)|\frac{3}{80}\rangle$, the two channels as $x \to 1$ corresponding to the $\frac{1}{10}$ and $\frac{3}{2}$ families. The solution corresponding to the single-block four-point function is easy to find, $f_0(x) = x^{-2/5}(x-1)^{4/5}$. The fact that f_0 is a solution of (C.8) determines the constant x_o in such a way that $\wp(z_o) = 0$. Factoring out the single block solution we arrive at a second order differential equation for the two blocks of $\langle\frac{3}{80}|\phi_{3/5}(x)\phi_{1/10}(1)|\frac{3}{80}\rangle$,

$$f'' + \frac{1}{5}\frac{13x-9}{x(x-1)}f' + \frac{3}{25}\frac{5x^2-10x+1}{[x(x-1)]^2}f = 0 \qquad (C.9)$$

36

its two solutions being expressed in terms of hypergeometric functions as follows,

$$f_{3/80}(x) \sim x^{-\frac{1}{5}}(x-1)^{-\frac{3}{5}}F(1/5, -1/5, 7/5, x) \qquad (C.10a)$$

$$f_{7/16}(x) \sim x^{-\frac{3}{5}}(x-1)^{-\frac{3}{5}}F(-3/5, -1/5, 3/5, x) \qquad (C.10b)$$

where the subscripts indicate the intermediate families in the channel $x \to 0$. Our previous analysis of one-point functions is also useful in order to determine the z-independent (but τ-dependent) normalizations of the three blocks. Let's label the three blocks in fig. 2 by $g_{1,2,3}$ respectively and we normalize them so that the residue of the leading singularity as $z \to 0$ is one. Then the correctly normalized blocks are,

$$G_1(z, \tau) = \eta(\tau)^{1/5}g_1(z, \tau), \ G_2(z, \tau) = \eta(\tau)^3 f_1(x)g_2(z, \tau) \qquad (C.11a)$$

$$G_3(z, \tau) = \eta(\tau)^3 f_2(x)g_3(z, \tau) \qquad (C.11b)$$

where $f_{1,2}(x)$ are given in (C.3) and x is related to τ via (3.1).

As the example above indicates nothing really different happens in the case where the two operators in the two-point function are not conjugate to each other.

Appendix D

In this appendix we will discuss certain techniques of solving FDE's on the torus for two-point functions. We will profit from the existence of integral representations for the two-point blocks. Such integral kernels can be written in terms of ϑ-functions on the torus.[*] However we will find more advantageous to use the fact that the torus can be represented as a branched sphere. Since the mapping function is the Weierstrass $\wp$-function this will be very convenient since the coefficient functions of (4.1) can be expressed naturally as rational functions of $\wp(z)$ and linear in $\wp'(z)$.

The map $w = \wp(z)$ maps the torus onto the w-sphere punctured at four points, $w = e_1, e_2, e_3, \infty$ with two branch-cuts connecting the four punctures. Here we use the standard notation,

$$e_1 = \frac{\pi^2}{3}[\vartheta_3^4 + \vartheta_4^4], \; e_2 = \frac{\pi^2}{3}[\vartheta_2^4 - \vartheta_4^4], \; e_3 = -\frac{\pi^2}{3}[\vartheta_2^4 + \vartheta_3^4] \qquad (D.1)$$

where ϑ_i are the usual elliptic ϑ-functions.[†] Using,

$$\wp'(z) = -2\left[(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)\right]^{\frac{1}{2}} \qquad (D.2a)$$

$$\frac{\partial}{\partial z} = -2[(w - e_1)(w - e_2)(w - e_3)]^{\frac{1}{2}}\frac{\partial}{\partial w} \qquad (D.2b)$$

$$\frac{\partial^2}{\partial z^2} = 4(w - e_1)(w - e_2)(w - e_3)\frac{\partial^2}{\partial w^2} + (6w^2 - \frac{g_2}{2})\frac{\partial}{\partial w} \quad etc. \qquad (D.2c)$$

one can transform an elliptic equation into a FDE on the sphere with regular singularities at e_1, e_2, e_3, ∞. Then, by a Moebius transformation, the singular points can be brought to $0, 1, \infty, x$, where x is defined by (3.1). Let ξ be the coordinate on the punctured sphere with the punctures at $0, 1, \infty, x$. Then the modular transformations act as follows, $S : \xi \to 1 - \xi$, $x \to 1 - x$, $T : \xi \to \xi/(\xi - 1)$, $x \to x/(x - 1)$. One the torus the only singular point of the equations was at $z = 0$. All the other singularities had to be apparent. This singular point on the sphere corresponds to $\xi = x$. The singularities at $\xi = 0, 1, \infty$ are generated from the branch cuts and the indices of the

---

[*]Representations of this form appeared in ref. [32-35] in connection with two-point functions of minimal models on the torus.

[†]We use the notation and conventions of [36].

equation there are fixed. For an n-th order equation the indices at $0, 1, \infty$ are $0, \frac{1}{2}, 1, \frac{3}{2}, \cdots, \frac{n}{2}$. The coefficients of the equation on the sphere as well as the location of some of the singularities generically depends on the modulus of the torus, however the monodromy data do not depend on it.

It is best to consider some concrete examples where the issues above can be easily understood. We will first consider the differential equation for the two blocks of the two-point function of the spin-$\frac{1}{2}$ representation of the $k = 1$ SU(2) WZW model. This was derived in [16],

$$\partial_z^2 f - \frac{3}{4}\wp(z)f = 0 \qquad (D.3)$$

Using (D.2) we can transform it into an equation in w the coordinate on the branched sphere,

$$\partial_w^2 f + \frac{1}{2}\left[\frac{1}{w - e_1} + \frac{1}{w - e_2} + \frac{1}{w - e_3}\right]\partial_w f - \frac{3}{16}\frac{w}{(w - e_1)(w - e_2)(w - e_3)}f = 0 \qquad (D.4)$$

We can now bring the singular points to $0, 1, \infty, x$ by the Moebius transformation,

$$\xi = \frac{(w - e_1)(e_3 - e_2)}{(w - e_2)(e_3 - e_1)} \qquad (D.5)$$

Equation (D.4) thus becomes,

$$\partial_\xi^2 f + \frac{1}{2}\left[\frac{1}{\xi} + \frac{1}{\xi - 1} + \frac{1}{\xi - x}\right]\partial_\xi f - \frac{1}{16}\frac{(2x - 1)\xi + x(x - 2)}{(\xi - x)^2\xi(\xi - 1)}f = 0 \quad (D.6)$$

This is now a standard FDE and by substituting $f = (\xi - x)^{-1/4}g$ we obtain a hypergeometric equation for $g^\ddagger$,

$$\xi(1 - \xi)g'' + \left[\frac{1}{2} - \xi\right]g' + \frac{1}{16}f = 0 \qquad (D.7)$$

which can be readily solved,

$$f_1(\xi) \sim (\xi - x)^{-\frac{1}{4}}\left[\sqrt{1 + \sqrt{\xi}} + \sqrt{1 - \sqrt{\xi}}\right] \qquad (D.8a)$$

$\ddagger$Since the difference of the two indices at $\xi = x$ is one this substitution is by itself enough to remove this singular point from the equation.

$$f_2(\xi) \sim (\xi - x)^{-\frac{1}{4}} \left[\sqrt{\sqrt{\xi - 1} + \sqrt{\xi}} - \sqrt{\sqrt{\xi - 1} - \sqrt{\xi}} \right] \qquad (D.8b)$$

As a next example we are going to discuss a differential equation that has a non-constant Wronskian. This is the one corresponding to the two point function of the 56-dimensional representation of the $k = 1$ E_7 WZW model. There are two blocks in this correlation function. The differential equation was derived in [16] and is the following,

$$\wp(z)\partial_z^2 f - \wp'(z)\partial_z f - \frac{3}{4}(\wp(z) + \frac{2}{3}g_2)f = 0 \qquad (D.9)$$

Transforming it on the sphere it becomes,

$$\partial_\xi^2 f + \frac{1}{2} \left[\frac{1}{\xi} + \frac{1}{\xi - 1} + \frac{3}{\xi - x} - \frac{4x - 2}{(2x - 1)\xi + x(x - 2)} \right] \partial_{xi} -$$

$$- \frac{1}{16} \frac{[(2x - 1)\xi + x(x - 2)]^2 + 8(x^2 - x + 1)(\xi - x)^2}{\xi(\xi - 1)((2x - 1)\xi + x(x - 2))(\xi - x)^2} f = 0 \qquad (D.10)$$

The singularity at $\xi = \frac{x(x-2)}{2x-1}$ is apparent with indices, (0,2) and is due to the zeros of the Weierstrass function in (D.9). By doing the transformation, $f(\xi) = (\xi - x)^{-3/4}g(\xi)$ (D.10) is transformed to,

$$\partial_\xi^2 g + \frac{1}{2} \left[\frac{1}{\xi} + \frac{1}{\xi - 1} - \frac{2}{\xi + \rho(x)} \right] \partial_\xi g + \frac{3}{16} \frac{\xi + \rho'(x)}{\xi(\xi - 1)(\xi + \rho(x))} g = 0 \quad (D.11)$$

where $\rho(x) = \frac{x(x-2)}{2x-1}$, $\rho'(x) = \frac{x(2-3x)}{2x-1}$. This can be solved by the same method that was used in the case of the presence of apparent singularities in section 3. It is in fact an easy task to show that if g is a solution to (D.11) and h is the solution to the hypergeometric equation,

$$\partial_\xi^2 h - \frac{1}{2} \left[\frac{3}{\xi} + \frac{1}{\xi - 1} \right] \partial_\xi h + \frac{1}{16} \frac{35\xi - 24}{\xi^2(\xi - 1)} h = 0 \qquad (D.12)$$

then,

$$g(\xi) = \partial_\xi h + \frac{6 - 5x}{4(x - 1)} \frac{1}{\xi} h \qquad (D.13)$$

The isomonodromic relation here is,

$$\frac{\partial g}{\partial x} = \frac{4\xi(\xi - 1)}{(2x - 1)\xi + x(x - 2)} \frac{\partial g}{\partial \xi} - \frac{(3x - 2)\xi - x}{(x - 1)((2x - 1)\xi + x(x - 2))} g \qquad (D.14)$$

Thus the two solutions of (D.11) are,

$$g_1(\xi, x) = \xi F(3/4, 1/4, 3/2, \xi) - \frac{x-2}{4(x-1)} F(-1/4, -3/4, 1/2, \xi) \quad (D.15a)$$

$$g_2(\xi, x) = \xi^{\frac{3}{2}} F(3/4, 5/4, 5/2, \xi) + \frac{x}{4(x-1)} \xi^{\frac{1}{2}} F(-1/4, 1/4, 3/2, \xi) \quad (D.15b)$$

As another example consider the two-point function of the G_2 theory, discussed in Appendix A. The equation on the sphere is,

$$\partial_\xi^3 f + \frac{3}{2} \left[\frac{1}{\xi} + \frac{1}{\xi-1} + \frac{1}{\xi-x} \right] \partial_\xi^2 f + \frac{3}{25} \frac{25\xi^2 - (36x+7)\xi + x(7x+11)}{\xi(\xi-1)(\xi-x)^2} \partial_\xi f +$$

$$+ \frac{39}{125} \frac{x(x-1)}{\xi(\xi-1)(\xi-x)^3} f = 0 \qquad (D.16)$$

which again can be solved by standard contour integrals.

References

1. A. A. Belavin, A. M. Polyakov, A. B. Zamolodchikov, Nucl. Phys. **B241** (1984) 333.

2. D. Friedan, Z. Qiu, S. Shenker, in "Vertex Operators in Mathematics and Physics", (J. Lepowsky, S. Mandelstam, I. M. Singer Eds. Springer, 1984).

3. E. Kiritsis, Phys. Lett. **B217** (1989) 427.

4. D. Friedan, S. Shenker, Nucl. Phys. **B281** (1987) 509.

5. G. Segal, to be published somewhere, sometime.

6. E. Verlinde, Nucl. Phys. **B300** (1988) 360; R. Dijkgraaf, E. Verlinde, Utrecht preprint, THU-88/25.

7. R. Brustein, S. Yankielowicz, J. B. Zuber, Tel-Aviv preprint TAUP-1647-88.

8. G. Moore, N. Seiberg, Phys. Lett. **B212** (1988) 451; Nucl. Phys. **B313** (1989) 16; IAS preprint, IASSNS-HEP-88/39.

9. R. Brustein, Y. Neeman, S. Sternberg, Tel-Aviv preprint, TAUP-1669-88.

10. K. Rehren, B. Schroer, Nucl. Phys. **B312** (1989) 715.

11. H. Sonoda, Nucl Phys. **B311** (1988) 401,417.

12. Vl. S. Dotsenko, V. A. Fateev, Nucl. Phys. **B240** [FS12] (1984) 312; ibid. **B251** [FS13] (1985) 691.

13. T. Eguchi, H. Ooguri, Phys. Lett. **B203** (1988) 44; Univ. of Tokyo preprint, UT-531.

14. J. Harvey, S. Naculich, Nucl. Phys. **B305** (1988) 417.

15. S. Mathur, S. Mukhi, Phys. Lett. **B210** (1988) 133.

16. S. Mathur, S. Mukhi, A. Sen, Nucl. Phys. **B305** (1988) 219; ibid. **B312** (1989) 15; Phys. Lett. **B213** (1988) 303; ibid. **B218** (1989) 176; Tata Institute preprint, TIFR/TH/88-50.

17. P. Christe, F. Ravanini, Nordita preprint, NORDITA-88/35P.

18. E. Kiritsis, Berkeley preprint, LBL-26384, UCB-PTH-88/32.

19. S. Panda, S. Rao, Phys. Lett. **B215** (1989) 530.

20. B. Blok, S. Yankielowicz, CERN preprint, CERN-TH-5250/88.

21. D. Lewellen, SLAC preprint, SLAC-PUB-4740.

22. S. Naculich, Brandeis preprint, BRX-TH-257.

23. P. Deligne, "Equations Differentielles à Points Singuliers Réguliers", Lecture Notes in Mathematics, Vol. 163, (Springer, 1970).

24. M. Yoshida, "Fuchsian Differential Equations", (Friedr. Vieweg & Sohn, 1987).

25. M. Ohtsuki, Tokyo J. Math. **5** (1982) 23.

26. S. Mathur, private communication.

27. A. B. Zamolodchikov, V. Fateev, Sov. J. Nucl. Phys. **43** (1986) 657).

28. T. Eguchi, H. Ooguri, Nucl. Phys. **B282** (1987) 308.

29. T. Kimura, Proc. Jap. Acad. **57** (1981) 285.

30. H. Kimura, K. Okamoto, Funk. Ekv. **26** (1983) 37.

31. Al. B. Zamolodchikov, Comm. Math. Phys. **96** (1984) 419.

32. J. Bagger, D. Nemeschansky, J. B. Zuber, Harvard preprint, HUTP-88/A051.

33. O. Foda, B. Nienhuis, Utrecht preprint, THU-88-34.

34. G. Felder, ETH preprint.

35. T. Jayaraman, K. S. Narain, CERN preprint, CERN-TH-5166/88.

36. A. Erdélyi et al. "Higher Transcendental Functions", McGraw-Hill, 1953.