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The Logarithm of the Derivative Operator and Higher Spin Algebras of W_∞ Type

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Abstract

We use the notion of the logarithm of the derivative operator to describe W_∞ type algebras as central extensions of the algebra of differential operators. We also provide closed formulae for the truncations of $W_{1+\infty}$ to higher spin algebras with $s \geq M$, for all $M \geq 2$. The results are extended to matrix valued differential operators, introducing a logarithmic generalization of the Maurer–Cartan cocycle.

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1. Introduction

The algebra of differential operators on a circle is becoming increasingly important in two dimensional physics, in particular the theory of conformal models with extended (higher spin) symmetries, the KP hierarchy of integrable differential equations and more recently in quantum gravity. Central extensions of this algebra provide a natural generalization of the Virasoro algebra which is generated by first order differential operators. It also contains the affine $U(1)$ current algebra generated by diffential operators of zero order.

¿From a mathematical point of view, the algebra of differential operators on S^1 describes a linear deformation of the algebra of divergence–free (or Hamiltonian) vector fields on $T^\star S^1$. This relation can be easily understood by applying Leibniz's rule to the commutator of two differential operators of order k and l; indeed, the result is an operator of order $k+l-1$ modulo lower order terms, which correspond to the deformation of the area preserving diffeomorphism algebra in question. In mathematical terms, the principal (leading) symbol of the commutator is the Poisson bracket of the principal symbols of the initial operators. The analogous description in quantum mechanics, in terms of Weyl ordered differential operators, is known as Moyal bracket [1].

Central extensions of the algebra of differential operators on S^1 have been considered only recently. In the physics literature the first results in this direction were obtained by considering the large N limit [2] of Zamolodchikov's W_N algebras [3]. The complete structure of W_∞, which was subsequently proposed by Pope, Romans and Shen [4] on a purely algebraic basis, has been established field theoretically in the context of parafermion models [5]. W_∞ and more generally $W_{1+\infty}$, which includes an additional $U(1)$ current in the spectrum, describe a central extension of the algebra of differential operators on S^1. The existence and uniqueness of such central extension was earlier established in the mathematics literature by various authors [6] (in the context of cyclic homology), thus generalizing the result of Gelfand and Fuchs for the Virasoro algebra [7].

In this paper we adopt the new concept of the logarithm of the derivative operator [8], which is very useful for defining the corresponding 2–cocycle and making the identification with the $W_{1+\infty}$ algebra mathematically elegant. Our work should be considered in this regard as providing a systematic description of the mathematical aspects of W_∞ type algebras in terms of a single object, namely $\log D$. This notion is introduced in section 2, following [8], using the calculus of pseudo–differential operators. In section 3 we construct a basis in the algebra of all differential operators which makes the identification with $W_{1+\infty}$ explicit. In this basis the $\log D$ cocycle diagonalizes, in the sense that it is non–zero only when the order of two differential operators is the same. We also present a closed formula for the truncation of $W_{1+\infty}$ to W_∞. These results are further extended in section 4 to higher spin truncations of $W_{1+\infty}$ with spectrum $s \geq M$, for all $M \geq 2$. In section 5 we consider the generalization to matrix valued differential operators and the $\log D$ generalization of the Maurer–Cartan cocycle. Finally, in section 6 we present our conclusions together with some ideas about the algebra of differential operators in more than one dimension.

2. The logarithm of the Derivative Operator

The ring $\mathcal{R}$ of pseudo–differential operators on a circle is the ring of formal series $A(x,D)=\sum_{-\infty}^{n} a_i(x)D^i$, where $a_i(x)\in C^\infty(S^1,k)$ with $k\in R,C$ and D corresponds to d/dx. The multiplication law in $\mathcal{R}$ is determined by the product of symbols

$$A(x,\xi)\circ B(x,\xi)=\sum_{k\geq 0}\frac{1}{k!}A_\xi^{(k)}(x,\xi)B_x^{(k)}(x,\xi), \tag{2.1a}$$

where

$$A_\xi^{(k)}(x,\xi)\equiv\sum_{i=-\infty}^{n} a_i(x)(\xi^i)^{(k)}\ ,\quad B_x^{(k)}(x,\xi)\equiv\sum_{i=-\infty}^{n} b_i^{(k)}(x)\xi^i \tag{2.1b}$$

and coincides with the usual multiplication law on the subalgebra $\mathcal{R}_+\subset\mathcal{R}$, consisting of differential operators which are polynomial in D. The notation used in eq.(2.1b) means that (k) is the k–th derivative of ξ^i and $b_i(x)$ respectively. The law (2.1) determines the Lie algebra structure on $\mathcal{R}$,

$$[A,B]=A\circ B-B\circ A. \tag{2.2}$$

There is also an operation $res:\mathcal{R}\to C^\infty(S^1)$ on the ring $\mathcal{R}$ defined by $res(\sum a_iD^i)=a_{-1}(x)$. The main property of the residue is $\int res[A,B]=0$, for any $A,B\in\mathcal{R}$ (here and below the integration is over the circle S^1).

We now consider the formal expression $\log D$. For any pseudo–differential operator $A\in\mathcal{R}$ the formal product $A\circ\log D$, according to eq.(2.1), where $\log\xi$ is the symbol of $\log D$, is certainly not contained in $\mathcal{R}$. The crucial point, however, is that the formal commutator $[\log D,A]=\log D\circ A-A\circ\log D$ belongs to $\mathcal{R}$. Thus, we define the action of $\log D$ on $\mathcal{R}$ by commutator, $[\log D,*]$. In coordinate form it is

$$[\log D,A]=\sum_{k\geq 1}\frac{(-1)^{k-1}}{k}A_x^{(k)}D^{-k}. \tag{2.3}$$

Note that even if A is a differential operator ($A\in\mathcal{R}_+$), the result $[\log D,A]$ is in general a pseudo–differential operator.

<u>THEOREM</u> [8]: *A non–trivial central extension of the Lie algebra $\mathcal{R}$ is given by the 2–cocycle*

$$\mathcal{C}(L,M)=\int res([L,\log D]\circ M)=\int res\left(\sum_{k\geq 1}\frac{(-1)^k}{k}L_x^{(k)}D^{-k}M\right), \tag{2.4}$$

where L and M are arbitrary pseudo–differential symbols on S^1. The restriction of this cocycle on $\mathcal{R}_+$ gives a non–trivial central extension of $\mathcal{R}_+$.

The restriction of this cocycle on the subalgebra of vector fields (i.e., first order differential operators) is the Gelfand–Fuchs cocycle of the Virasoro algebra. Indeed,

$$\begin{aligned}\mathcal{C}(f(x)D,g(x)D)&=\int res([f(x)D,\log D]\circ g(x)D)\\&=\int res((-f'(x)D^0+f''D^{-1}/2-f'''D^{-2}/3+\ldots)g(x)D)\\&=\int res(\ldots+f'''(x)g(x)D^{-1}/6+\ldots)\\&=\frac{1}{6}\int f'''(x)g(x)dx,\end{aligned} \tag{2.5}$$

which implies the non–triviality of the cocycle on $\mathcal{R}$ and $\mathcal{R}_+$.

For the proof, the skew symmetry of $\mathcal{C}(L,M)$ follows immediately from the identities

$$[\log D, LM] = [\log D, L]M + L[\log D, M] \quad \text{and} \quad \int res[\log D, A] = 0, \tag{2.6}$$

for any $L, M, A \in \mathcal{R}$. These identities themselves are consequences of eqs.(2.1)–(2.3). The same identities, together with the Jacobi identity on $\mathcal{R}$, allow the verification of the cocycle property,

$$\sum_{cyclic} \mathcal{C}(L,[M,N]) = \int res([L,\log D][M,N] + [N,\log D][L,M] + [M,\log D][N,L]) = 0. \tag{2.7}$$

The value of $\mathcal{C}(f(x)D^m, g(x)D^n)$ on the homogeneous generators of $\mathcal{R}$ vanishes for $n+m+1<0$, but in general it does not vanish for $m+n+1\geq 0$. The restriction of this cocycle on differential operators ($n,m\geq 0$) coincides with the formula [9],[10]

$$\mathcal{C}(f(x)D^m, g(x)D^n) = \frac{m!n!}{(m+n+1)!}\int f^{(n)}g^{(m+1)}dx. \tag{2.8}$$

As we mentioned, this central extension of the algebra of differential operators on S^1 is unique [6] (up to a multiplicative constant).

In the remaining part of this section we describe another basis in $\mathcal{R}$, in which the action of $[\log D, *]$ becomes simple. For this we remind that the algebra $\mathcal{R}$ carries a natural conjugacy operation $* : (\sum_i a_i(x)D^i)^* = \sum_i (-1)^i D^i a_i(x)$. We also recall that an arbitrary pseudo–differential operator is a sum of self–adjoint and skew self–adjoint operators. A basis for self–adjoint operators is $\{D^m f(x) D^m\}$, where m is integer, while skew self–adjoint operators have odd degree and can not be written in this form. For them we consider the same expression $D^m f(x) D^m$, with half–integer m. Even though fractional powers of D do not belong in $\mathcal{R}$, the above expression defines a pseudo–differential operator. To verify this we only have to rewrite $D^m f D^m$ in the canonical form $\sum a_j D^j$, applying the commutation relation and observing that all fractional powers disappear at the end. We also note that $\log D$ is a self–adjoint operator (more precisely, we consider $\log|D|$ which is an even function of D) and the commutator $[\log D, A]$ changes the parity of A.

If we now consider the value of the logarithmic 2–cocycle on the generators in this basis, we find that

$$\mathcal{C}(D^m f(x) D^m, D^n g(x) D^n)\ , \text{ where } m,n \text{ are integer or half-integer} \tag{2.9}$$

depends on the sum $m+n$, but it does not depend on the particular choice of m and n. Indeed,

$$\begin{aligned}\mathcal{C}(D^m f(x) D^m, D^n g(x) D^n) &= \int res([D^m f(x) D^m, \log D] D^n g(x) D^n) \\ &= \int res(D^m [f(x), \log D] D^m D^n g(x) D^n) \\ &= \int res([f(x), \log D] D^{m+n} g(x) D^{m+n}).\end{aligned} \tag{2.10}$$

It also vanishes for $m+n+1<0$.

In the next section we will describe yet another basis in which the 2–cocycle becomes diagonal, being non–zero only if the order of two differential operators is the same. The latter is more natural from the point of view of conformal field theory, in view of the explicit identification we would like to establish with $W_{1+\infty}$. The dictionary that one should keep in mind is that the order of a differential operator is equal to $s-1$, where s is the spin (conformal dimension) of the corresponding W–generator.

3. $W_{1+\infty}$, W_∞ and $\log D$

The starting point in this section is the $W_{1+\infty}$ algebra whose generators are denoted by V^s_m, with $m\in Z$ and $s\in Z^+$. Following [11], it is convenient to introduce the notation

$$g_l^{ss'}(m,n;\mu)=\frac{\varphi_l^{ss'}(\mu)}{2(l+1)!}N_l^{ss'}(m,n)\ , \tag{3.1}$$

where

$$\varphi_l^{ss'}(\mu):=\sum_{k=0}^{\infty}\frac{\left(-\frac{1}{2}-2\mu\right)_k\left(\frac{3}{2}+2\mu\right)_k\left(-\frac{l+1}{2}\right)_k\left(-\frac{l}{2}\right)_k}{k!\left(-s+\frac{3}{2}\right)_k\left(-s'+\frac{3}{2}\right)_k\left(s+s'-l-\frac{3}{2}\right)_k}\ , \tag{3.2}$$

$$\begin{aligned}N_l^{ss'}(m,n):=\sum_{k=0}^{l+1}(-)^k\binom{l+1}{k}(2s-l-2)_k[2s'-k-2]_{l+1-k}\cdot\\ \cdot[s-1+m]_{l+1-k}[s'-1+n]_k\end{aligned} \tag{3.3}$$

and

$$(a)_k:=a(a+1)\cdots(a+k-1)\ \ ,\ \ [a]_k=a(a-1)\cdots(a-k+1). \tag{3.4}$$

Then, the commutation relations of $W_{1+\infty}$ are given by

$$\begin{aligned}[V^s_m,V^{s'}_n]=((s'-1)m-(s-1)n)\,V^{s+s'-2}_{m+n}+c_s(m;\mu)\delta_{s,s'}\delta_{m+n,0}+\\ +\sum_{r\geq 1}g_{2r}^{ss'}(m,n;\mu)V^{s+s'-2-2r}_{m+n}\ ,\end{aligned} \tag{3.5}$$

with $\mu=-\frac{1}{2}$. The central term is

$$c_s(m;-\frac{1}{2})=c\frac{(m+s-1)!}{(m-s)!}\frac{2^{2(s-3)}[(s-1)!]^2}{(2s-1)!!(2s-3)!!}. \tag{3.6}$$

The series of subleading terms with $r\geq 1$ terminates with either V^2_{m+n} or V^1_{m+n}, depending on whether $s+s'$ is even or odd respectively. This is reflected in the form of the hypergeometric function (3.2) and the other combinatorial factors (3.3). They both have zeros (complementing each other), which guarantee the termination of the series (3.5) at V^2 or V^1 for $\mu=-\frac{1}{2}$. $W_{1+\infty}$ contains the Virasoro algebra as a subalgebra, generated by $\{V^2_m\}$. The conformal dimension of all other generators V^s is s, as follows from the commutation relations $[V^2_m,V^s_n]$.

It is more convenient in the sequel to introduce $z=e^{ix}$ and work with Laurent series in z instead of trigonometric functions in x. We will also denote ∂_z by D and use contour integration for the definition of the $\log D$ cocycle. Having set up the notation, we now consider differential operators $\{z^{m+s-1}D^{s-1}\}$ on S^1 with $m \in Z$, $s \in Z^+$. In order to make the identification between the two algebras explicit, we introduce the basis

$$V^s_m = -B(s) \sum_{k=1}^{s} \alpha^s_k \binom{m+s-1}{k-1} z^{m+s-k} D^{s-k}, \tag{3.7}$$

where

$$B(s) = \frac{2^{s-3}(s-1)!}{(2s-3)!!} \; ; \; \alpha^s_k = \frac{(2s-k-1)!}{[(s-k)!]^2}. \tag{3.8}$$

<u>THEOREM</u>: *The differential operators (3.7) satisfy the commutation relations (3.5) with* $c = 0$. *The value of the* $\log D$ *cocycle in this basis is given by*

$$\mathcal{C}(V^s_m, V^{s'}_n) = -\frac{B(s)^2}{2s-1} \frac{(m+s-1)!}{(m-s)!} \delta_{s,s'} \delta_{m+n,0}. \tag{3.9}$$

This establishes the desired result, the advantage of the basis (3.7) being that the 2–cocycle vanishes unless $s = s'$.

We note that the diagonal basis so constructed is natural from the point of view of conformal field theory, since the corresponding local quantum field theoretic operators $V^s(z) := \sum_{m \in Z} V^s_m z^{-m-s}$ are quasi–primary (i.e., highest weight with respect to the $SL(2,R)$ subalgebra of the Virasoro algebra.) Moreover, the $SL(2,R)$ Ward identities, which reflect the respective invariance of the vacuum, imply that $\langle V^s(z) V^{s'}(w) \rangle \sim \delta_{s,s'}$. In this basis, the generators (3.7) are not self–adjoint (as differential operators) for $s \geq 3$; if that were the case, the coefficients of the subleading terms in eq.(3.5) would be identical to those of the Moyal bracket algebra, which assumes a hermitian (e.g., Weyl) ordering for the differential operators. To obtain the value of the cocycle (3.6), we have to multiply $\mathcal{C}$ by a constant $-c$, c being the central charge of the Virasoro subalgebra

$$[V^2_m, V^2_n] = (m-n) V^2_{m+n} + \frac{c}{12}(m^3 - m) \delta_{m+n,0}. \tag{3.10}$$

Next, we consider the algebra W_∞ whose generators $\{W^s_m \; ; \; s \geq 2, \; m \in Z$ satisfy the commutation relations

$$[W^s_m, W^{s'}_n] = ((s'-1)m - (s-1)n) W^{s+s'-2}_{m+n} + c_s(m;\mu) \delta_{s,s'} \delta_{m+n,0}$$

$$+ \sum_{r \geq 1} g^{ss'}_{2r}(m,n;\mu) W^{s+s'-2-2r}_{m+n}, \tag{3.11}$$

with $\mu = 0$ and

$$c_s(m;0) = \frac{c}{2} \frac{(m+s-1)!}{(m-s)!} \frac{2^{2(s-3)} s!(s-2)!}{(2s-1)!!(2s-3)!!}. \tag{3.12}$$

The complete structure of W_∞ , which arises as the large N limit of Zamolodchikov's W_N algebras [2], [5], was proposed by Pope, Romans and Shen [4]. It resembles the structure of $W_{1+\infty}$, but since $\mu = 0$ now, the series of terms with $r \geq 1$ automatically terminates at W^2_{m+n} or W^3_{m+n}, depending on whether $s + s'$ is even or odd respectively. We also note that the normalization of the central charges (3.12) is different from (3.6).

The W_∞ algebra can be obtained from $W_{1+\infty}$ by truncation to $s \geq 2$, provided that W^s_m are expressed in terms of V^s_m as

$$W^s_m = V^s_m + \frac{B(s)}{s-1}\sum_{l=1}^{s-1}(-)^l \frac{(2s-2l-1)}{B(s-l)}\frac{(m+s-1)!}{(m+s-l-1)!}V^{s-l}_m \tag{3.13}$$

for all $s \geq 2$. Of course, from the point of view of the algebra of all differential operators on S^1, the truncation to $s \geq 2$ can be done automatically, using a basis that does not contain operators of zero order. We point out, however, that the log D cocycle will not maintain its diagonal form, if we naively use the basis (3.7), after subtracting the $k = s$ terms. Therefore, in order to accomodate the commutation relations of W_∞ into the present framework, we have to construct a different basis.

In terms of differential operators, the appropriate basis can be found by combining eqs.(3.7), (3.8) and (3.13). The result we find is

$$W^s_m = -\frac{B(s)}{s-1}\sum_{k=1}^{s-1}\beta^s_k \binom{m+s-1}{k-1} z^{m+s-k}D^{s-k}, \tag{3.14}$$

with

$$\beta^s_k = \frac{(2s-k-1)!}{(s-k)!(s-k-1)!}. \tag{3.15}$$

A theorem analogous to the previous one also holds for the W_∞ algebra (3.11), but in this case the value of the log D cocycle for the operators (3.14) is

$$\mathcal{C}(W^s_m, W^{s'}_n) = \frac{B(s)B(s+1)}{2(s-1)}\frac{(m+s-1)!}{(m-s)!}\delta_{s,s'}\delta_{m+n,0}. \tag{3.16}$$

To obtain arbitrary values for the central charge of the Virasoro subalgebra, we simply have to multiply $\mathcal{C}$ by the numerical factor $c/2$.

The general relation between the algebra of diferential operators on S^1 and W_∞ has also been addressed by Fairlie and Nuyts [12]. Using the theory of Moyal brackets they found, among other things, a basis of operators which yield the structure constants $g^{ss'}_l$ of W_∞ . In this regard, some of the closed formulae we have presented here should be considered as being complementary to theirs.

4. Higher Spin Truncations of $W_{1+\infty}$

The algebra of differential operators on S^1 can be truncated from below by considering only elements with order bigger or equal than a fixed positive integer. This procedure leads to higher spin algebras with spectrum $s \geq M$, for all $M \in Z^+$. The truncation from above, on the other hand, is not possible while maintaining the linear structure of the algebra. A truncation method of the second type has been discussed by Radul and Vaysburd [13], who proposed a systematic description of W_N algebras as factor algebras of $\mathcal{R}_+$ and of its central extension. In this section we focus on truncations of the first type and construct bases in which the log D cocycle becomes diagonal for all M. This is a natural generalization of the results described in the previous section. Such higher spin algebras have been discussed before in a different context [14] and clearly, for all

$M \geq 3$, they do not contain a Virasoro subalgebra. Hence, although consistent with the Jacobi identity, their meaning in quantum field theory is obscure, if these algebras are supposed to represent all the symmetries of a chiral two dimensional model. They are interesting, nevertheless, from a mathematical point of view. It might also turn out that these algebras admit a natural interpretation in the framework of higher dimensional field theories, but this point of view is still lacking.

THEOREM: *For any $M \in Z^+$, the subalgebra of differential operators with order bigger or equal than $M-1$ admits a non–trivial central extension, given by the restriction of the* $\log D$ *cocycle. In the basis*

$$W_m^s = -\frac{(s-M)!}{(s-1)!} B(s) \sum_{k=1}^{s-M+1} \frac{(2s-k-1)!}{(s-k)!(s-k-M+1)!} \binom{m+s-1}{k-1} z^{m+s-k} D^{s-k} \quad (4.1)$$

the $\log D$ *cocycle is diagonal.*

In this basis, the commutation relations of the corresponding higher spin algebra assume the form (3.11), with $\mu = (M-2)/2$. The series of subleading terms terminates in the general case with either W^M or W^{M+1} and, of course, the overall normalization of the central terms also depends on M.

This result implies equivalently a relation between the W generators of the higher spin algebra with $s \geq M$ and those of $W_{1+\infty}$ of the form $W_m^s = V_m^s +$ *lower spin V–terms*, which is analogous to eq.(3.13). We will present this relation in a field theoretic form, for arbitrary M, using the standard realization of $W_{1+\infty}$ [15] in terms of a complex free fermion ψ (and its conjugate $\bar{\psi}$) in two dimensions. We have

$$W^s(z) := \sum_{m \in Z} W_m^s z^{-m-s} = B(s) \frac{(s+M-2)!(s-M)!}{[(s-1)!]^2} \cdot$$

$$\cdot \sum_{k=0}^{s-M} (-1)^k \binom{s-1}{k} \binom{s-1}{k+M-1} \partial^k \psi(z) \partial^{s-1-k} \bar{\psi}(z), \quad (4.2)$$

for all $s \geq M$. The higher spin truncations of $W_{1+\infty}$ become obvious in this realization, since for fixed M, the operator product expansion of any two fields (4.2) generates in its singular terms only fields with spin bigger or equal than M.

5. Colored W_∞ and Matrix $\log D$

In this section we consider the algebra of matrix valued differential operators and study its central extension using a logarithmic generalization of the Maurer–Cartan cocycle. We will show that for the unitary group $U(p)$, this extension reproduces the colored W_∞^p algebra [16] and more generally the non–abelian current version of $W_{1+\infty}$ [17].

Let $\mathcal{G}$ be a reductive matrix Lie algebra. In analogy with section 2, we consider the space of pseudo–differential operators on the circle with matrix coefficients, i.e., $A(x, D) = \sum_{i=-\infty}^{n} a_i(x) D^i$ with $a_i \in C^\infty(S^1, \mathcal{G})$. The same multiplication law (2.1) now includes not only the usual Leibniz rule, but also the matrix product of the coefficients. These operators form an associative and hence Lie algebra $\mathcal{R}_\mathcal{G}$. An operation res : $\mathcal{R}_\mathcal{G} \to C^\infty(S^1, \mathcal{G})$ is naturally defined on this algebra by the trace of the coefficient of the D^{-1} term, $res(\sum a_i(x)D^i) = tr[a_{-1}(x)]$. The action of $\log D$ on $\mathcal{R}_{\mathcal{G}}$ is given by the same formula

$$[\log D, A(x,D)] = \sum_{k\geq 1} \frac{(-1)^{k-1}}{k} A_x^{(k)} D^{-k}, \tag{5.1}$$

as before, but now the symbol of $\log D$ is the matrix $(\log \xi)\cdot 1_{\mathcal{G}}$, where $1_{\mathcal{G}}$ is the unit matrix.

THEOREM: *For an arbitrary reductive matrix Lie algebra $\mathcal{G}$, the algebra of pseudo–differential operators $\mathcal{R}_{\mathcal{G}}$ has a non–trivial central extension given by the 2-cocycle*

$$\mathcal{C}(A,B) = \int res\left([A,\log D]\circ B\right). \tag{5.2}$$

The cocycle property of $\mathcal{C}(A,B)$ can be verified in the same way as in section 2. The algebra $\mathcal{R}_{\mathcal{G}}$ contains the subalgebra of the zeroth order differential operators $\{a(x)D^0\}$, which is naturally isomorphic to the corresponding loop algebra $\tilde{\mathcal{G}}$. It is easy to see that the restriction of $\mathcal{C}(A,B)$ on this subalgebra gives the Maurer-Cartan cocycle $\int tr[a'b]$ and thus defines the corresponding affine (centrally extended current) algebra $\hat{\mathcal{G}}$. This remark implies by itself the non–triviality of the 2–cocycle and that of the central extension of $\mathcal{R}_{\mathcal{G}}$.

In analogy with the scalar case, we may restrict ourselves to the subalgebra $\mathcal{R}^+_{\mathcal{G}}$ of all matrix differential operators, $\{\sum_{i=0}^n a_i D^i\}$ and show that $\mathcal{R}^+_{\mathcal{G}}$ has a non–trivial central extension. It turns out that this specific central extension for $\mathcal{R}^+_{U(p)}$ (differential operators with unitary coefficients) coincides with the colored $W^p_{1+\infty}$ algebra [17] and with W^p_∞ [16], when truncated to spin $s\geq 2$. It is straightforward to extend the results of the previous sections to this case and construct a basis in which the cocycle becomes diagonal and the commutation relations of the algebra assume the form

$$[V^s_{a,m}, V^{s'}_{b,n}] = ((s'-1)m-(s-1)n)(\delta^{ab}V^{s+s'-2}_{0,m+n} + d^{abc}V^{s+s'-2}_{c,m+n})+$$

$$+c_s(m;\mu)\delta^{ab}\delta^{s,s'}\delta_{m+n,0} + \sum_{r\geq 1} g^{ss'}_{2r}(m,n;\mu)(\delta^{ab}V^{s+s'-2-2r}_{0,m+n} + d^{abc}V^{s+s'-2-2r}_{c,m+n})-$$

$$-\frac{1}{4}f^{abc}(V^{s+s'-1}_{c,m+n} + 2\sum_{r\geq 1} g^{ss'}_{2r-1}(m,n;\mu)V^{s+s'-1-2r}_{c,m+n}), \tag{5.3a}$$

$$[V^s_{0,m}, V^{s'}_{\alpha,n}] = ((s'-1)m-(s-1)n)V^{s+s'-2}_{\alpha,m+n} + c_s(m;\mu)\delta^{\alpha,0}\delta^{s,s'}\delta_{m+n,0}+$$

$$+\sum_{r\geq 1} g^{ss'}_{2r}(m,n;\mu)V^{s+s'-2-2r}_{\alpha,m+n}, \tag{5.3b}$$

with group indices $\alpha = (0,a)$, $a = 1,\ 2\cdots p^2-1$. f^{abc} are the structure constants of the $SU(p)$ subgroup of $U(p)$ and d^{abc} is the third order completely symmetric Casimir tensor (which vanishes for $SU(2)$.) As before, $\mu = -\frac{1}{2}$ for $W^p_{1+\infty}$, $\mu = 0$ for W^p_∞ and so on for the colored higher spin truncations of the algebra.

The scalar $W_{1+\infty}$ algebra is contained in (5.3) as the $U(1)$ (trace) part of $U(p)$. We also note that the truncations of $\mathcal{R}^+_{\mathcal{G}}$ from below do not contain the current algebra $\tilde{\mathcal{G}}$ and thus, the corresponding restrictions of the $\log D$ do not include the Maurer–Cartan cocycle. One can show, however, that the restriction of the $\log D$ cocycle on any such truncation is still non–trivial [18], as in the scalar case.

6. Discussion

In this paper we have presented the mathematical aspects of W_∞ type algebras, using the notion of the logarithm of the derivative operator [8] and its matrix generalizations. This notion provides a systematic way to describe central extensions of the algebra of all differential operators on the circle and establish the isomorphism with $W_{1+\infty}$.

We would also like to point out that in this context, the connection of the $W_{1+\infty}$ algebra and the KP hierarchy (see for instance [14]) is not surprising. For zero central charge, $W_{1+\infty}$ becomes the pure algebra of all differential operators, which is dual to integral operators defining the phase space of the KP equations.

The log D cocycle admits a natural generalization to higher dimensional compact manifolds $\mathcal{M}$. This was constructed by Radul [19], using Wodzicki's residue formula [20]. To emphasize the significance of this problem we remind that for higher dimensional manifolds there is no invariant decomposition of pseudo–differential operators X into purely differential and integral parts, $X = X_+ + X_-$. Hence, the knowledge of a residue formula is very important for defining central extensions of the algebra of (pseudo)–differential operators on $\mathcal{M}$. The latter is closely related to the structure of the cotangent bundle of $\mathcal{M}$ and to the group of symplectomorphisms of $T^\star\mathcal{M}$, since the principal symbol of a differential operator naturally defines a Hamiltonian flow in $T^\star\mathcal{M}$. For any pseudo–differential operator X on $\mathcal{M}$, $resX$, which was originally introduced by Wodzicki in the framework of spectral geometry, is unique and defines a trace functional. For $\mathcal{M} \simeq S^1$ the usual residue formula is recovered. With this ingredient, the $\log D$ cocycle is naturally generalized to all compact manifolds.

The field theoretical aspects of the resulting infinite dimensional algebras have not been studied for general $\mathcal{M}$. This could justify further work on the subject, both from the physical and mathematical point of view.

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