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General Virasoro Construction on Affine g

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Abstract

We study the general Virasoro construction $L = L_{ab} \, {}^*_* J^a J^b \, {}^*_*$ on the currents of affine **g**, obtaining the master equation for the inverse inertia tensor L_{ab}. Sugawara and coset constructions are only the simplest solutions of this system, as illustrated here by a class of generalized spin-orbit constructions with generically irrational central charge.

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Affine Lie algebras were discovered independently in mathematics [1] and physics [2]. The first representations [2] were constructed with world-sheet fermions [2,3] to implement the proposal of current-algebraic spin and internal symmetry on the string [2]. Examples of affine-Sugawara constructions [2,4] and coset constructions [2,4] were also given in the first string era, as well as the vertex operator construction of fermions and $SU(N)_1$ from compactified spatial dimensions [5]. The group-theoretic generalization of these constructions [6,7,8] and their application to the heterotic string [9] mark the beginning of the present era. See [10-14] for further remarks.

Less familiar is the original spin-orbit* construction [2,15] studied in parallel with the early coset constructions, which has remained for 18 years as an example of a class of conformal constructions more general than Sugawara and coset constructions.

Motivated by the spin-orbit construction and recent consideration of Virasoro constructions with arbitrary (2,0) operators [16], we study the general Virasoro construction on the currents J^a of affine $\mathbf{g}$

$$L_m = L_{ab} \, {}^*_* J^a J^b \, {}^*_{*m} \;\; , \;\; [L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0} \tag{1}$$

with symmetric normal-ordering ${}^*_* J^a J^b {}^*_* = {}^*_* J^b J^a {}^*_*$, obtaining the master equation for the inverse inertia tensor $L_{ab} = L_{ba}$. Remarks on the most general quadratic form including linear terms in J^a are found in the concluding paragraph. The master equation contains at least Sugawara, coset and generalized spin-orbit constructions, the latter being distinguished by generically irrational central charge.

We begin our construction with the algebra of affine $\mathbf{g}$ [1,2]

$$J^a(z)J^b(w) = \frac{G^{ab}}{(z-w)^2} + if^{ab}{}_c \left[\frac{1}{(z-w)} + \frac{1}{2}\partial \right] J^c(w) + T^{ab}(w) + O(z-w) \tag{2.a}$$

$$[J^a_m, J^b_n] = if^{ab}{}_c J^c_{m+n} + G^{ab} m \delta_{m+n,0} \;\; , \;\; m,n \in Z \tag{2.b}$$

for Lie algebra $\mathbf{g}$ not necessarily semi-simple or compact with structure constants $f^{ab}{}_c$. To obtain level $x_i = 2k_i/\psi_i^2$ of $\mathbf{g}_i$ in $\mathbf{g} = \oplus_i \mathbf{g}_i$ take

$$G^{a(i)b(j)} = \delta^{ij} k_i g_i^{ab} \;\; , \;\; T^a = \oplus_i T_i^a \tag{3}$$

where g_i^{ab} is a Killing metric of $\mathbf{g}_i$ and $(T^a)_c^b = -if^{ab}{}_c$ is the adjoint of $\mathbf{g}$ with $Tr(T_i^a T_i^b) = Q_i g_i^{ab}$ and $\tilde{h}_{g_i} = Q_i/\psi_i^2$. The quadratic Casimirs in the adjoint Q_i of the non-compact

*Affine-Sugawara and coset constructions were originally called additive or spin-spin interactions [2,4] among the new currents J^g on the string, since they had no interaction terms with the orbital operators $\partial\phi$ of spacetime. String physics today is additive in the original sense. In order to introduce new spin-gauges, the non-additive or spin-orbit construction [2,15] also coupled the spin currents $J^{G/H}$ of $G/H{=}SO(N-1,2)/SO(N-1,1)$ (for any level of N=4 [2] and level one of any N [15]) to the orbital operators in the form $\partial\phi J^{G/H}$.

generalizations of $\mathbf{g}_i$ are the same as in the compact case and we choose ψ_i to be the highest root of the compact group, so the dual Coxeter numbers $\tilde{h}_{g_i}$ are also the same for the compact and non-compact cases. The OPE (2.a) defines the symmetric normal-ordered current bilinear $T^{ab}(z) = T^{ba}(z)$,

$$T^{ab}_m = \sum_{n\in Z} {}^*_* J^a_{m+n} J^b_{-n} {}^*_* = \sum_{n>-m} J^b_{-n} J^a_{m+n} + \sum_{n<-m} J^a_{m+n} J^b_{-n} + \frac{1}{2}[J^a_0, J^b_m]_+ + \frac{i}{2} f^{ab}{}_c m J^c_m \quad (4)$$

which satisfies $(T^{ab}_m)^\dagger = T^{ab}_{-m}$ when J^a is hermitian and

$$\langle 0|T^{ab}(z)J^c(w)|0\rangle = 0 \quad , \quad T^{ab}_{m\geq -1}|0\rangle = 0 \quad (5)$$

where $|0\rangle$ is the $SL(2,R)$-invariant vacuum defined by $J^a_{m\geq 0}|0\rangle = 0$.

We then compute

$$T^{ab}(z)J^c(w) = M^{ab,c}{}_d \left[\frac{1}{(z-w)^2} + \frac{1}{(z-w)}\partial + \frac{1}{2}\partial^2\right] J^d(w) +$$

$$+ N^{ab,c}{}_{de}\left[\frac{1}{(z-w)} + \frac{3}{4}\partial\right] T^{de}(w) + W^{abc}(w) + O(z-w) \quad (6.a)$$

$$M^{ab,c}{}_d = \frac{1}{2}(f^{bc}{}_e f^{ea}{}_d + f^{ac}{}_e f^{eb}{}_d) + G^{ac}\delta^b_d + G^{bc}\delta^a_d \quad (6.b)$$

$$N^{ab,c}{}_{de} = \frac{i}{2}\left[(\delta^a_d f^{bc}{}_e + \delta^a_e f^{bc}{}_d) + (a \leftrightarrow b)\right] \quad (6.c)$$

which defines the (3,0) operator $W^{abc}(z)$ and

$$T^{ab}(z)T^{cd}(w) = \frac{P^{ab,cd}}{(z-w)^4} + Q^{ab,cd}{}_e\left[\frac{1}{(z-w)^3} + \frac{1}{2(z-w)^2}\partial + \frac{1}{6(z-w)}\partial^2\right] J^e(w) +$$

$$+ R^{ab,cd}{}_{ef}\left[\frac{1}{(z-w)^2} + \frac{1}{(z-w)}\partial\right] T^{ef}(w) + S^{ab,cd}{}_{efg}\frac{W^{efg}(w)}{(z-w)} + O(1) \quad (7)$$

among the bilinears. The coefficients in (7) are

$$P^{ab,cd} = (\frac{1}{2} f^{bc}{}_e f^{ad}{}_f G^{ef} + G^{ac}G^{bd}) + (a \leftrightarrow b) \quad (8.a)$$

$$\begin{aligned} Q^{ab,cd}{}_e =& i\left[(G^{ac}f^{bd}{}_e + G^{bc}f^{ad}{}_e) + (c \leftrightarrow d)\right] + \frac{i}{4}\Big[(f^{af}{}_e f^{bd}{}_g f^{cg}{}_f + \\ & + f^{af}{}_e f^{bc}{}_g f^{dg}{}_f + f^{cf}{}_e f^{ad}{}_g f^{bg}{}_f + f^{df}{}_e f^{ac}{}_g f^{bg}{}_f) + (a \leftrightarrow b)\Big] \end{aligned} \quad (8.b)$$

$$R^{ab,cd}{}_{ef} = (R_1 - R_2 - R_3)^{ab,cd}{}_{ef} \quad (8.c)$$

$$R_1^{ab,cd}{}_{ef} = \frac{1}{2}\left[G^{ac}(\delta^b_e\delta^d_f + \delta^d_e\delta^b_f) + G^{ad}(\delta^b_e\delta^c_f + \delta^c_e\delta^b_f)\right] + (a \leftrightarrow b) \quad (8.d)$$

$$R_2^{ab,cd}{}_{ef} = \frac{1}{2}(f^{ac}{}_e f^{bd}{}_f + f^{ad}{}_e f^{bc}{}_f) + (a \leftrightarrow b) \quad (8.e)$$

$$R_3^{ab,cd}{}_{ef} = \frac{1}{4}\Big[\delta_f^d(f^{ac}{}_g f^{bg}{}_e + f^{bc}{}_g f^{ag}{}_e) + \delta_f^c(f^{ad}{}_g f^{bg}{}_e + f^{bd}{}_g f^{ag}{}_e) + \\ +\delta_f^a(f^{cb}{}_g f^{dg}{}_e + f^{db}{}_g f^{cg}{}_e) + \delta_f^b(f^{ca}{}_g f^{dg}{}_e + f^{da}{}_g f^{cg}{}_e)\Big] + (e \leftrightarrow f) \tag{8.f}$$

$$S^{ab,cd}{}_{efg} = i\Big[\delta_e^b(\delta_f^d f^{ac}{}_g + \delta_f^c f^{ad}{}_g) + \delta_f^b(\delta_e^d f^{ac}{}_g + \delta_e^c f^{ad}{}_g) + \\ +\delta_g^b(\delta_f^d f^{ac}{}_e + \delta_f^c f^{ad}{}_e) + \delta_f^b(\delta_g^d f^{ac}{}_e + \delta_g^c f^{ad}{}_e) + \\ +\delta_e^b(\delta_g^d f^{ac}{}_f + \delta_g^c f^{ad}{}_f) + \delta_g^b(\delta_e^d f^{ac}{}_f + \delta_e^c f^{ad}{}_f)\Big] + (a \leftrightarrow b) \tag{8.g}$$

so that, in particular, Q and S are antisymmetric under $ab \leftrightarrow cd$ while P and R are symmetric.

We now focus on $L(z) = L_{ab}T^{ab}(z)$ with L_{ab} the inverse inertia tensor, for which

$$[L_m, J_n^a] = -n\Big[2G^{ab}L_{be} + f^{ab}{}_d L_{bc} f^{cd}{}_e\Big] J_{m+n}^e - 2if^{ab}{}_d L_{bc} T_{m+n}^{cd} \tag{9}$$

is obtained from (6). The Virasoro algebra (1) for L_m follows with (7) for any solution L_{ab} of the master equation

$$2L_{ab} = L_{cd}L_{ef}R^{cd,ef}{}_{ab} \ , \ \ c = 2G^{ab}L_{ab} = 2Tr(GL) \tag{10}$$

alternate forms of which include

$$L_{ab} = 2L_{ac}G^{cd}L_{db} - L_{cd}L_{ef}f^{ce}{}_a f^{df}{}_b - L_{cd}f^{ce}{}_f(L_{ea}f^{df}{}_b + L_{eb}f^{df}{}_a) \tag{11.a}$$

$$L_{ab}^{ij} = 2\sum_l (L^{il}k_l g_l L^{lj})_{ab} + L_{cd}^{ij}L_{ef}^{ij}(T_i^c)_a^e(T_j^d)_b^f + [L_{cd}^{jj}L_{ea}^{ji}(T_j^c T_j^d)_b^e + (i,a \leftrightarrow j,b)] \tag{11.b}$$

where $L_{a(i)b(j)} \equiv L_{ab}^{ij}$ are the $(\mathbf{g}_i, \mathbf{g}_j)$ blocks of L_{ab} and $c = 2\sum_i k_i Tr(g_i L^{ii})$. The simple form of the central charge in (10) is obtained with the help of the master equation.

We discuss a number of properties of the master equation (10-11).

(A) The Sugawara construction

$$L^g \ : \ \ (L^g)_{ab}^{ij} = (g_i)_{ab}\frac{\delta^{ij}}{2k_i + Q_i} \ , \ \ c_g = \sum_i \frac{x_i dim\mathbf{g}_i}{x_i + \tilde{h}_{g_i}} \tag{12}$$

is always a solution, and similarly for L^h when $\mathbf{g} \supset \mathbf{h}$.

(B) K-conjugate pairs [2,4]: If L is a solution, so is $K \equiv L^g - L$ with $c(K) = c_g - c(L)$. This pairing was shown for an arbitrary conformal construction in [16] and follows in the present case because

$$U_{cd}L_{ef}^g R^{cd,ef}{}_{ab} = 2U_{ab} \tag{13}$$

is an identity for any symmetric matrix U. Moreover, $[L_m, K_n] = 0$ follows with (13) and the further identity $(L^g)_{ab}Q^{ab,cd}{}_e = 0$. The coset constructions [2,4,8] $K = L^g - L^h$ are recovered for $L = L^h$.

(C) There are no solutions infinitesimally close to Sugawara since any such deformation δL about L^g must satisfy $\delta L_{ab}{=}\delta L_{cd} L^g_{ef} R^{cd,ef}{}_{ab}{=}2\delta L_{ab}$ according to (10) and (13), so that $\delta L_{ab} = 0$. Conformal deformations about Sugawara and coset constructions by linear terms in J^a are studied in [12,17,18].

(D) There are no deformations $\delta L = \epsilon L^g$, $\epsilon \ll 1$ in the Sugawara direction about any solution L since $\delta L_{ab} = \epsilon L^g_{ab} {=} \epsilon L^g_{cd} L_{ef} R^{cd,ef}{}_{ab} {=} 2\epsilon L_{ab}$ according to (10), (13), and $L = L^g/2$ is not a solution.

(E) We have examined the case $\mathbf{g} = su(2)$ with $G^{ab} = k\delta^{ab}$ and $a, b = 1, 2, 3$ in detail since the inverse inertia tensor can be diagonalized to $L_{ab} = \lambda_a \delta_{ab}$ by a transformation in G. The solutions of the resulting equations

$$\lambda_a(1 - 2k\lambda_a)\delta_{ab} = 2\sum_{c,d} \lambda_c(\lambda_a + \lambda_b - \lambda_d)\epsilon_{cda}\epsilon_{cdb} \tag{14}$$

coincide with the Sugawara and coset constructions.

We turn now to generalized spin-orbit constructions on $\mathbf{u}(1)^{D_{g/h}} \oplus \mathbf{g}$, $\mathbf{g} \supset \mathbf{h}$ with the ordering $J^a{=}(\pi^I \equiv i\partial\phi^I, J^I, J^A)$ where $I = 1, \cdots, D_{g/h}$, $A = 1, \cdots, D_h$ and D_g, D_h, $D_{g/h}$ are the dimensions of $\mathbf{g}$, $\mathbf{h}$, $\mathbf{g/h}$ respectively. The simplest spin-orbit ansatz is

$$L_{ab} = \begin{pmatrix} g_{IJ} \begin{pmatrix} \varepsilon\alpha & \beta \\ \beta & \gamma \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \end{pmatrix} & g_{AB}\delta \end{pmatrix} \ , \ G^{ab} = k \begin{pmatrix} \varepsilon g^{IJ} & 0 & 0 \\ 0 & g^{IJ} & 0 \\ 0 & 0 & g^{AB} \end{pmatrix} \tag{15}$$

with $\varepsilon = \pm 1$ for simple $\mathbf{g}$ and $\mathbf{h}$, which corresponds to Virasoro operators $L = \alpha\pi^2 {+} 2\beta\pi \cdot J^{g/h} {+} \gamma(J^{g/h})^2 {+} \delta(J^h)^2$. Consistency of this ansatz in the master equation (10-11) requires that G/H be a symmetric space, so that $f^{IKA} f_{JKA} = \delta^I_J Q_g/2$ and $f^{AIJ} f_{BIJ} = (Q_g - Q_h)\delta^A_B$, which includes $SU(N)/SO(N)$, $SU(2N)/Sp(N)$, $SO(N{+}1)/SO(N)$, $F_4/Spin(9)$, $E_6/Sp(4)$, E_6/F_4, $E_7/SU(8)$, $E_8/SO(16)$ and their non-compact generalizations. In what follows we refer without loss of generality only to compact $\mathbf{g}$, since all results are the same for the non-compact generalizations. The restriction to a symmetric space follows from the requirement that the OPE of the spin-orbit term $\pi \cdot J^{g/h}$ with $(J^h)^2$ reproduces the simple spin-orbit term, and not a matrix generalization. In fact, the ansatz (15) with $\delta_1(J^{h_1})^2 {+} \delta_2(J^{h_2})^2$ and the results below hold also for $\mathbf{h} = \mathbf{h}_1 \oplus \mathbf{h}_2$, $\mathbf{h}_1 = \mathbf{h}_2$, $Q_{h_1} = Q_{h_2} = Q_h$ and $\delta_1 = \delta_2 = \delta$ which includes $SO(2N)/SO(N) \otimes SO(N)$, $Sp(2N)/Sp(N) \otimes Sp(N)$ and $G_2/SU(2) {\otimes} SU(2)$ as well. More general $\mathbf{h} \subset \mathbf{g}$ will require more complicated matrix couplings.

The resulting equations with $\bar{\beta}^2 = \varepsilon\beta^2$

$$\alpha = 2k(\alpha^2 + \bar{\beta}^2) \ , \ \bar{\beta} = \bar{\beta}(2k(\alpha + \gamma) + \frac{1}{2}(\gamma + \delta)Q_g) \tag{16.a}$$

$$\gamma = (2k + Q_g)\gamma^2 + 2k\bar{\beta}^2 \ , \ \delta = (2k + Q_h)\delta^2 + \gamma(2\delta - \gamma)(Q_g - Q_h) \tag{16.b}$$

$$c = 2k((\alpha+\gamma)D_{g/h} + \delta D_h) \tag{16.c}$$

exhibit only Sugawara and coset solutions when $\beta = 0$. For $\beta \neq 0$ we obtain after some algebra the spin-orbit solutions

$$\alpha = \frac{1}{4k}\Big(1 + \eta F^{-1}(8k + 4Q_h - 3Q_g)\Big) \tag{17.a}$$

$$\beta = \tilde{\eta} F^{-1}\sqrt{(-\varepsilon/k)(4k + 2Q_h - Q_g)} \tag{17.b}$$

$$\gamma = \frac{1 - \eta F^{-1}(8k + 4Q_h - Q_g)}{2(2k + Q_g)} \tag{17.c}$$

$$\delta = \frac{1 + \eta F^{-1}(5Q_g - 4Q_h)}{2(2k + Q_g)} \tag{17.d}$$

$$F = \sqrt{(3Q_g - 4Q_h)^2 - 32k(Q_g - Q_h)} \tag{17.e}$$

with η, $\tilde{\eta}= \pm 1$. The corresponding central charges

$$c = \frac{1}{2}\left[\frac{xD_g}{x+\tilde{h}_g} + D_{g/h}\right] + \frac{\eta\left[x(5\tilde{h}_g - 4\tilde{h}_h)D_h + \tilde{h}_g(2x + 4\tilde{h}_h - 3\tilde{h}_g)D_{g/h}\right]}{2\tilde{F}(x+\tilde{h}_g)} \tag{18.a}$$

$$\tilde{F} = \sqrt{(3\tilde{h}_g - 4\tilde{h}_h)^2 - 16x(\tilde{h}_g - \tilde{h}_h)} \tag{18.b}$$

are generically irrational for each of the 4 solutions $L(\eta,\tilde{\eta})$. Among these the K-conjugate pairs are identified with $L(\eta,\tilde{\eta}) + L(-\eta,-\tilde{\eta}) = L^g$ while $L_\pm(\eta) = L(\eta,\tilde{\eta}=\pm 1)$ for each η are pairs with the same central charge. The latter pairing, corresponding to a sign change of the spin-orbit term, is an accidental degeneracy due to the abelian nature of $\mathbf{g}_1$ or $\mathbf{g}_2$, since the corresponding symmetry $L^{12} \to -L^{12}$ in the master equation follows only when the second term on the right of (11.b) vanishes.

A restriction to real central charge puts an upper bound on the level

$$x < \frac{(3\tilde{h}_g - 4\tilde{h}_h)^2}{16(\tilde{h}_g - \tilde{h}_h)} \tag{19}$$

which also implies reality of the coefficients α, γ and δ. Under this condition, the levels of the exceptional groups are limited to $x \le 2$ for $E_6/Sp(4)$, $x \le 3$ for $E_7/SU(8)$, $x \le 4$ for $E_8/SO(16)$ and there are no solutions for $F_4/Spin(9)$, E_6/F_4 and $G_2/SU(2)\otimes SU(2)$. Reality of the spin-orbit coefficient β for $x \ge 1$ also requires that $\varepsilon = -1$. This means that the spin-orbit constructions are not explicitly unitary, although it may be possible to find unitary subspaces [2,15,19]. Indeed, Mandelstam [15] showed that the spin-orbit construction for level one of $SO(9,2)/SO(9,1)$ with $c = 28, -25/2$ is equivalent to the NS model [20] in 10 dimensions.† We have also checked that the central charges

†More generally, one has $g^{IJ} = \left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$ for the original construction [2,15] on $SO(N-1,2)/SO(N-1,1)$, so that $\varepsilon g^{IJ} = \left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right)$ for the abelian group is the ordinary Minkowski metric on N spacetime dimensions.

come in positive-negative pairs for $\eta = \pm 1$ when $\xi \equiv 4\tilde{h}_h - 3\tilde{h}_g > 0$, which includes $SO(N+1)/SO(N)$, but there are cases with both central charges positive when $\xi < 0$, e.g. level one of $SU(2N)/Sp(N)$, $N \geq 8$.

The generalized spin-orbit solutions above should be considered as conformal constructions with couplings $J^{h_i} J^{g_j/h_j}$, $i \neq j$ where $\mathbf{h}_i$ is an abelian subgroup of $\mathbf{g}_i$ such that $D_{h_i} = D_{g_j/h_j}$. The spin-orbit couplings are distinct from the $J^{g_i} J^{g_j}$ interactions obtainable in coset constructions. A systematic solution of the master equation, say at large level, and/or a geometric interpretation of its form is clearly desirable.

We also remark that the OPE's (6,7) can be easily applied to the most general quadratic form $L_m = L_{ab} T^{ab}_m + D_a(m) J^a_m$+constant. As an example, the c-changing but $SL(2,R)$-preserving case [12,17,18] $L(z) = L_{ab} T^{ab}(z) + D_a \partial J^a(z)$ gives

$$2L_{ab} = L_{cd} L_{ef} R^{cd,ef}{}_{ab} + 2\Delta_{ab} \ , \ D_a(2G^{ab} L_{be} + f^{ab}{}_d L_{bc} f^{cd}{}_e) = D_e \qquad (20.a)$$

$$\Delta_{ab} = i(L_{ac} f^{ce}{}_b + L_{bc} f^{ce}{}_a) D_e \ , \ c = 2G^{ab}(L_{ab} - \Delta_{ab} - 6D_a D_b). \qquad (20.b)$$

Conformal deformations by the (1,0) currents of Sugawara and coset constructions [12,17,18] solve this system for arbitrary D_a with $\Delta_{ab} = 0$, although reality of the central charge and L_{ab} is also guaranteed with D_a imaginary. The c-fixed deformations [12,17,18] $L_m = L_{ab} T^{ab}_m + D_a J^a_m + \frac{1}{2} G^{ab} D_a D_b$ maintain the original master equation and the central charge in (10-11), requiring only the additional eigenvalue condition in (20.a) on D_a.

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