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Proof of the Completeness of the Classification of Rational Conformal Theories with c = 1*

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Abstract

By using the Serre-Stark theorem we prove that the list of rational conformal theories with c = 1, given by Ginsparg is in fact complete.

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It has been a major goal of Conformal Field Theory (CFT) to classify all the universality classes of critical behavior in two dimensions. Stated from the point of view of string theory, this would correspond to a classification of the classical ground states of the string. There has been a lot of progress in recent years in understanding the tools to achieve such a classification, and some partial results have emerged. Despite the fact that a complete classification seems out of reach at present, some partial classifications may shed light on which ingredients are important and which are not.

Ginsparg, in a nice paper [1], gave a list of c=1 theories. There are two continuous families in this list corresponding to the torus and $\mathbf{Z}_2$ orbifold models. In [2], it was shown that there are no marginal deformations that lead outside these two lines. In [1], three disconnected models have been constructed, that could not be reached from the rest via marginal deformations. There was a wide-spread suspicion that his list was complete. In fact, in ref. [3], it was shown that under the assumption that the building blocks are toroidal partition functions, then this list is complete.

In this paper we will supplement this classification by a rigorous proof that the subset of rational CFT's at c=1 is complete.

In order to set the notation, we will write the partition functions of these models. For the torus models, which are parametrized by the radius R of the target space, the partition function is

$$\mathbf{Z}(R) = \frac{1}{n\bar{n}} \sum_{m,n \in \mathbf{Z}} q^{\frac{1}{2}P_L^2} \bar{q}^{\frac{1}{2}P_R^2} , \tag{1}$$

where $P_L = \frac{m}{R} + \frac{nR}{2}$, $P_R = \frac{m}{R} - \frac{nR}{2}$. The partition function (1) is modular invariant, and is also invariant under the duality transformation $R \to \frac{2}{R}$. Define $\mathbf{Z}_N \equiv \mathbf{Z}(N\sqrt{2})$. Then the orbifold partition function at radius R is given by:

$$\mathbf{Z}(R)_{\rm orb} = \frac{1}{2}\left(\mathbf{Z}(R) + 2\mathbf{Z}_2 - \mathbf{Z}_1\right) . \tag{2}$$

The disconnected models which are in one-to-one correspondence with the three non-trivial discrete subgroups of SU(2) have the following partition functions:

$$\mathbf{Z}_T = \frac{1}{2}\left(2\mathbf{Z}_3 + \mathbf{Z}_2 - \mathbf{Z}_1\right) \quad , \quad \mathbf{Z}_0 = \frac{1}{2}\left(\mathbf{Z}_4 + \mathbf{Z}_3 + \mathbf{Z}_2 - \mathbf{Z}_1\right)$$

$$\mathbf{Z}_1 = \frac{1}{2}\left(\mathbf{Z}_5 + \mathbf{Z}_3 + \mathbf{Z}_2 - \mathbf{Z}_1\right) . \qquad (3).$$

The ratiional theories belonging to this list are the ones which contain partition functions with R being the square root of a rational number.

The partition function of a CFT with central charge c behaves as $\tau \to i\infty$ as in ref. [4]:

$$\mathbf{Z}(\tau\,,\bar{\tau}) \longrightarrow \left[e^{2ni(\tau-\bar{\tau})}\right]^{-\frac{c}{24}}\,\left(1+0\left(e^{2ni\tau\epsilon}e^{-2ni\bar{\tau}\bar{\epsilon}}\right)\right) \tag{4}$$

with $\epsilon\,,\bar{\epsilon} > 0$.

In a rational conformal theory the partition function can be constructed out of modular forms invariant under a finite index subgroup of the modular group, [5]. The modular forms have to have the appropriate behavior as $\tau \to i\infty$ as indicated in eq. (4). In particular for c=1 they have to be what mathematicians call one-singular forms. Fortunately the one-singular modular forms have been classified by Serre and Stark, [6].

Define:

$$f_{a,b} = \sum_{n\,\in\,\mathbf{Z}} q^{\alpha(n+\frac{b}{2a})^2}\ ,\ g_{a,b} = \sum_{n\,\in\,\mathbf{Z}}(-1)^n\ q^{\alpha(n+\frac{b}{2a})^2}\,, \tag{5}$$

with $a,b\epsilon\frac{1}{2}\mathbf{Z}\,,\alpha > 0$ and as usual $q = e^{2\pi i\tau}$, $f_{a,b(\tau)}$ satisfied the following peridicity properties in b:

$$f_{a,b}(\tau) = f_{a,-b}(\tau) = f_{a,2a+b}(\tau) = f_{a,2a-b}(\tau)\,. \tag{6}$$

Thus for each $\alpha\epsilon\mathbf{Z}$ the "fundamental domain" in $b\epsilon\mathbf{Z}$ is $0 \le b \le a$.

The modular properties of $f_{a,b}$ and $g_{a,b}$ are as follows:

$$f_{\alpha,\beta}(-1/\tau) = \sqrt{\frac{-i\tau}{2\alpha}}\sum_{k=0}^{2\alpha-1} e^{i\pi kb/\alpha}\begin{cases} f_{alpha,k} & \text{if } b\in\mathbf{Z}\\ g_{\alpha,k(\tau)} & \text{if } b\in\mathbf{Z}+1/2\end{cases} \tag{7}$$

$$g_{\alpha,\beta}(-1/\tau) = \sqrt{\frac{-i\tau}{2\alpha}}\sum_{k=0}^{2\alpha-1} e^{i\pi(2k+1)b/\alpha}\begin{cases} f_{alpha,k+1/2} & \text{if } b\in\mathbf{Z}\\ g_{\alpha,k+1/2} & \text{if } b\in\mathbf{Z}+1/2\end{cases} \tag{7'}$$

$$f_{\alpha,beta}(\tau+1) = \begin{cases} e^{i\pi ub^2/2\alpha}f_{alpha,\beta(\tau)} & \text{if } \alpha-b\in\mathbf{Z}\\ e^{i\pi b^2/2\alpha}g_{\alpha,\beta(\tau)} & \text{if } \alpha-b\in\mathbf{Z}+1/2\end{cases} \tag{8}$$

and similarly for $g_{\alpha,\beta}$. Thus $F_{\alpha,\beta} \equiv \frac{f_{\alpha,\beta}}{\eta}$ where η is the Dedkind n-function are modular forms for some $\Gamma(N)$ and if $\alpha-\beta\in\mathbf{Z}$ then $N = (4\alpha,24)$.*

⋆ By (α,β) we mean the lowest common multiple of $\alpha, beta$.

Thus at this point we can invoke the Serre–Stark theorem, [6], which states that any 1-singular modular form on some $\Gamma(N)$ is a finite linear combination of the functions $F_{\alpha,\beta}(\tau)$. That any modular form of a finite index subgroup Γ of the modular group can be constructed out of $F_{\alpha,\beta}$ can be inferred from the following. If c, h are rational then there is an integer (positive) N such that $T^N = \mathbf{1}$ on the representation. Thus the representation can be shown to be invariant under some $T(N)$ with N large enough.

Thus the general partition function for a rational $c = 1$ CFT can be written up to overall normalization as

$$\mathbf{Z}(\tau,\bar{\tau}) = \sum_{\substack{(\alpha,\beta)\\ \bar{\alpha},\bar{\beta}}} N_{\alpha,\beta}^{\bar{\alpha},\bar{\beta}} F_{\alpha,\beta}(\tau)\bar{F}_{\bar{\alpha},\bar{\beta}}(\bar{\tau}) \tag{9}$$

There are two basic constraints on (9). The first is modular invarinace and the second is that all multiplicities should be non-negative integers when one normalizes the partition function so that the unit operator appears once.†

The first step will be to construct all possible modular invariants of the form (9). We call α in $F_{\alpha,\beta}(\tau)$ the "level." From (7) it is obvious that $b \in \mathbf{Z}$ and from (8) that $\alpha \in \mathbf{Z}^+$. Since the action of S and T does not change the level of the modular form $F_{\alpha,\beta}(\tau)$ we can classify the invariants at a given level separately.

Lemma 1: Let $\alpha \in \mathbf{Z}^+$. Then for any divisor δ of α there exists an invariant $\mathbf{Z}^{\alpha}_{\delta}$. The invariant generated by $\bar{\delta} = \alpha/\delta$ is the same. The form of the invariant is given by:

$$\mathbf{Z}^{\alpha}_{\delta} = \sum_{k=0}^{\delta-1}\sum_{\lambda=0}^{\bar{\delta}-1}\{F_{\alpha k\bar{\delta}+\lambda\delta(\tau)}\bar{F}_{\alpha,k\bar{\delta}-\lambda\delta(\bar{t}au)}$$

$$+F_{\alpha,\alpha+k\bar{\delta}+\lambda\delta(\tau)}\bar{F}_{\alpha,\alpha+k\bar{\delta}-\lambda\delta(\bar{\tau})}\} \tag{10}$$

Proof: The line of arguments follows the discussion in [7]. To save space we will just sketch the proof. Invariance under T implies that $b^2 - b'^2 = 0\mathrm{mod}(4a)$ for the term $F_{\alpha,\beta(\tau)}\bar{F}_{\alpha,\beta(\bar{\tau})}$ to appear in the partition function. Choosing δ such that $\delta|\alpha$, $\bar{d}elta = \alpha/\delta$ we see that $(b-b')(b+b') = 0\mathrm{mod}(4\delta\bar{\delta})$, and the solution is $b\delta m + \bar{\delta}n$, $b' = \delta m - \bar{\delta}n$, m, n integers. One then

† There is an extra constraint and this is that the theory has to have a consistent operator algebra. We will return to this point later on.

by using the arguments in [5] can show that these solutions for every divisor δ of a generate the commutant of T. We can then check that the partition function invariant under T is also invariant under S by using (7). Restricting b, b' in the domain $0 \leq b, b' \leq 2a - 1$ gives the invariant (10). It is obvious that $\mathbf{Z}_{\delta}^{\alpha} = \mathbf{Z}_{\bar{d}elta}^{\alpha}$. The above shows that the invariant $\mathbf{Z}_{\delta}^{\alpha}$ exhaust all possible invariants. Not all of them are independent though.

Lemma 2: Let $\alpha = \alpha' p^2$ where p is a prime greater than one. Then for every divisor δ of α' we have $\mathbf{Z}_{\delta p}^{\alpha' p^2} = \mathbf{Z}_{\delta}^{\alpha'}$.

Proof: The proof is obvious once we establish the following identity:

$$\sum_{k=0}^{N-1} F_{\alpha N^2, N(2\alpha k+b)(\tau)} = F_{\alpha,\beta(\tau)} \tag{11}$$

which follows trivially from the definition (5).

Theorem: Let $\alpha \in \mathbf{Z}^+$. Let δ be a divisor of α, $\bar{d}elta = \alpha/\delta$ such that the greatest common divisor $[\delta, \bar{\delta}] = 1$ For all such pairs $\delta, \bar{\delta}$ there exists an independent invariant $\mathbf{Z}_{\delta}^{\alpha}$ given by (10). Moreover, this list of invariants is complete.

Proof: Let δ be a divisor of α such that $[\delta, \bar{\delta}] > 1$ then by Lemma 2 this invariant can be reduced to an invariant at a lower level $\alpha' = \alpha/[\delta, \bar{\delta}]^2$. At a given level invariant for different pairs $(\delta, \bar{\delta})$ and $(\delta', \bar{\delta}')$ are distinct by inspection. Furthermore it is easy to show that relation (11) is the only relation between forms at different levels. Completeness follows from Lemma 1.

It is obvious that the invariant $\mathbf{Z}_{\delta}^{\alpha}$ is equal to the toroidal partition function $\mathbf{Z}(R)$ (see (1)) at $R = \sqrt{\delta/2\bar{\delta}}$. Duality is the statement that $\mathbf{Z}_{\delta}^{\alpha} = \mathbf{Z}_{\bar{\delta}}^{\alpha}$.

Thus we have proven that an arbitrary modular invariant partition function is a finite linear combination of the invariants $\mathbf{Z}_{\delta}^{\alpha}$ (toroidal partition functions). The final step is to impose the constraint of non-negative integer multiplicities. This question has been answered in ref. [3]. In order to make the line of thought clear we will present the statements without proofs. Details fo the proofs will appear in [8].

We will distinguish two cases. The first case is when no chiral (1,0) operator appears in the partition function. In this case there is no $U(1)$ symmetry and we have to write partition functions in terms of Virasoro characters.

The general partition function can be written as a finite sum $\mathbf{Z} = \sum_{i=1}^{N} c_i \mathbf{Z}(R_i)$, $\sum_{i=1}^{N} c_i = 1$, in order to have a unique (0,0) operator.

Lemma 3: The multiplicity of $(s^2, 0)$ operators, $s \in \mathbf{Z}^+$, $s \in s$ is given by $1 + 2[s/N]$ for $\mathbf{Z}_N$ and is 1 otherwise. In particular $\mathbf{Z}_1$ contains 3(1,0) operators and any other $\mathbf{Z}(R_i)$ only one.

The constraint that there are no (1,0) operators in the partition function forces $c_1 = -1/2$.

Lemma 4: For all i, $zc_i \in \mathbf{Z}$. This is so because we can show that in the opposite case there will always be fractional multiplicities in $\mathbf{Z}$.

Lemma 5: Except for $c_1 = -1/2$ all other c_i have to be zero or positive. This is because we can show that if one of them is negative there will be at least one state that appears with a minus sign in $\mathbf{Z}(\tau, \bar{\tau})$.

Theorem: Any partition function $\mathbf{Z}(\tau, \bar{\tau})$ not containing any (1,) operators has to be either of the form (2) or of the form (3).

This follows for Lemmas 3, 4, 5 and the constraint that operators of dimension $(s^2, 0)$, $s \leq 5$ have positive integer multiplicities. The only case left to consider is the case that there exist (1,0) operators in the theory, so that states are in representations of a chiral $U(1)$ algebra. If the number of operators (1,0) is $N_1 > 0$ then c_1 (the coefficient of $\mathbf{Z}$) has to be $N_1 - 1/2$. Lemma 4 still remains true for the same reason, and Lemma 5 extends trivially in the statement that $c_1 \geq 0 \forall i$. From the above it is obvious that there are only two possible solutions, either the torus models $\mathbf{Z} = \mathbf{Z}(R)$ or $\tilde{\mathbf{Z}} = 1/2(\mathbf{Z}(R) + \mathbf{Z}(R'))$. The models with partition function $\tilde{Z}$ do not have a consistent operator algebra. In particular they do not satisfy the requirement of additively conserved $U(1)$ changes. Thus they do not correspond to CFT models.

Thus our claim is proven. There are no other rational CFT's with $c = 1$ except those in [1].

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