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Non-standard Bosonization Techniques in Conformal Field Theory*

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Abstract

It is shown that G/H models can be constructed in terms of a number of free bosons with a stress energy tensor that contains vertex operators. Generalizations of this technique are also discussed.

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The major goal of Conformal Field Theory (CFT), is to classify all universality classes of critical phenomena in $d = 2$, and to find exact solutions. At the moment in the absence of a concrete and easy to work with plan of classification, attempts point to the study of special classes of theories, with the hope that their features will give a handle on the general problem. An interesting class of theories are rational conformal theories. The list known today is composed of G/H models [1], and rational toroidal models along with the associated orbifolds. There are two questions which are of interest in this respect. First, are there other rational conformal theories not in the list above? Second, how can we obtain an exact solution for all such theories?

This paper is considered as a continuation of ref. [2]. In [2] it was shown how one could use a scalar field to describe an Ising model. The basic idea was to use a modified stress-energy tensor for the scalar field. In the present paper we are going to discuss how one could generalize this technique to an arbitrary G/H model. We will also indicate the most general framework, that this technique can work.

We are going to discuss extensively a specific example, namely $SU(2)$ at level N and the associated $\mathbf{Z}_N$ parafermionic theory. One way to proceed is to construct $SU(2)_N$ as a diagonal subgroup of a product of N copies of $SU(2)_1$. $SU(2)_1$ can be constructed out of one free boson.‡ So we consider N free bosons $\phi^i, i = 1, 2, \dots, N$ and we normalize their 2-point functions as usual.§

$$\langle 0|\phi_i(z)\phi_j(w)|0\rangle = -\delta^{ij}\log(z-w) \tag{1}$$

We can construct the $SU(2)_1$ currents in the usual way (the subscript indicates different $SU(2)$'s)

$$J^3_j(z) = \frac{i}{\sqrt{2}}\partial_z\phi_j(z), \quad J^\pm_j(z) \equiv \frac{1}{\sqrt{2}}(J^1_j(z) \pm iJ^2_j(z)) = \frac{1}{\sqrt{2}}e^{\pm i\sqrt{2}\phi_j(z)} \quad j = 1, 2, \dots, N \tag{2}$$

The diagonal $SU(2)$ has level N and its currents are diagonal sums of the individual $SU(2)_1$ currents.

$$J^3(z) = \frac{i}{\sqrt{2}}\sum_{j=1}^{N}\partial_z\phi_j(z), \quad J^\pm(z) = \frac{1}{\sqrt{2}}\sum_{j=1}^{N}e^{\pm i\sqrt{2}\phi_j(z)} \tag{3}$$

It is straightforward to compute the Sugawara stress-tensor in terms of the currents in (3). The

‡ In all of our discussions, by free bosons we mean bosons without background charge.

§ From now on we will focus on the holomorphic part of the theory.

answer is:

$$T_N(z) = \frac{1}{N+2}\left[-\frac{1}{2}\left(\sum_{i=1}^{N}\partial_z\phi_i\right)^2 - \sum_{i=1}^{N}(\partial_z\phi_i)^2 + \sum_{i<j}^{N}\left(e^{i\sqrt{2}(\phi_i-\phi_j)} + e^{-i\sqrt{2}(\phi_i-\phi_j)}\right)\right] \quad (4)$$

To make things more transparent we will make a transformation on the basic variables ϕ_i. We define:

$$\phi_i = \frac{\Phi}{\sqrt{N}} + \sqrt{2}\vec{\mu}_i \cdot \vec{\rho} \quad (5)$$

where ρ is an $(N-1)$-dimensional vector of scalar fields. The $\vec{\mu}_i$ are the fundamental weights of $SU(N)$. They are $(N-1)$-dimensional vectors and there are N of them. They are normalized as follows:

$$\vec{\mu}_i \cdot \vec{\mu}_j = -\frac{1}{2N} + \frac{1}{2}\delta^{ij} \quad (6)$$

The roots of $SU(N)$ are $\vec{\alpha}_{ij} = \vec{\mu}_i - \vec{\mu}_j$. They are normalized to $\vec{\alpha}\cdot\vec{\alpha} = 1$. The simple roots are $\vec{\alpha}_i = \vec{\mu}_i - \vec{\mu}_{i+1}$. The new basis of fields defined in (5) is still orthonormal.

$$\langle 0|\rho_i(z)\rho_j(w)|0\rangle = -\delta^{ij}\log(z-w), \quad \langle 0|\Phi(z)\Phi(w)|0\rangle = -\log(z-w) \quad (7)$$

In terms of the new variables, the $SU(2)$ currents become

$$J^3(z) = i\sqrt{\frac{N}{2}}\partial_z\Phi, \quad J^{\pm}(z) = \frac{1}{\sqrt{2}}e^{\pm i\sqrt{2/N}\Phi}\left[\sum_{i=1}^{N} e^{\pm 2i\vec{\mu}_i\cdot\vec{\rho}}\right] \quad (8)$$

whereas the stress-energy tensor is

$$T_N(z) = -\frac{1}{2}(\partial_z\Phi)^2 + T_N^{par}(z), \quad T_N^{par}(z) = -\frac{1}{N+2}\left[\partial_z\vec{\rho}\cdot\partial_z\vec{\rho} + \sum_{\vec{\alpha}} e^{2i\vec{\alpha}\cdot\vec{\rho}}\right] \quad (9)$$

The sum is over all roots of $SU(N)$.[¶] From (8) one can easily identify the parafermion operators:

$$\psi_{\pm 1}(z) = \frac{1}{\sqrt{N}}\left[\sum_{i=1}^{N} e^{\pm i\sqrt{2}\vec{\mu}_i\cdot\vec{\rho}}\right] \quad (10)$$

One then by using the OPE among vertex operators can construct the rest of the parafermion

¶ The reason for the appearance of the roots of $SU(N)$ is the fact that the $SU(2)_N$ parafermions can be constructed as the coset model $SU(N)_1 \otimes SU(N)_1/SU(N)_2$.

operators explicitly and verify their operator algebra*

$$\psi_k(z)\psi_{k'}(w) = C_{k,k'}(z-w)^{-2kk'/N}[\psi_{k+k'}(w) + O(z-w)] \tag{11}$$

$$\psi_k(z) = \binom{N}{k}^{-1/2} \sum_{i_1<i_2<\ldots<i_k}^{N} e^{i\sqrt{2}(\vec{\mu}_{i_1}+\vec{\mu}_{i-2}+\ldots\vec{\mu}_{i_k})\cdot\vec{\rho}}$$

$$C^2_{k,k'} = \frac{(k+k')!(N-k)!(N-k')!}{k!k'!(N-k-k')!N!}$$

The spin fields of the parafermions are non-local with $\psi_k(z)$. Their effect is to twist the vertex operators that define the parafermions. Unfortunately a more concrete description of them is still lacking. The expectation is that they correspond to the twist fields of some orbifold of the scalar theory compactified on the root-lattice of $SU(N)$.

This is exactly what happens in the case $N = 2$ which was worked out explicitly in [2]. There the spin field of the Ising model is one of the two twist fields of the $\mathbf{Z}_2$ orbifold model. The $\mathbf{Z}_2$ orbifold can be though in this case as the orbifold with respect to the Weyl group of $SU(2)$. This might imply that the appropriate orbifold models in the general N case would be the ones obtained by twisting by some subgroup of the Weyl group of $SU(N)$.

By this construction one can also obtain the partition function of the parafermionic theory. The technique is a direct generalization of the one used in [2]. We must compute the expectation value of T_N^{par} on the torus, and then integrate it to obtain the characters. A straightforward computation shows that the partition function of parafermions obtained that way are of the form:

$$\mathbf{Z}_{\vec{\alpha},\vec{\beta}}(\tau) = \eta(\tau)\left[\frac{\theta_{\vec{\alpha},\vec{\beta}}(\tau)}{\eta(\tau)^(N-1)}\right]^{2/(N+2)} \tag{12}$$

where $\eta(\tau)$ is the Dedkind n-function, $\vec{\alpha}$ is a vector on the $SU(N)$ root lattice and $\vec{\beta}$ appropriate vectors on L^*/L. To verify that (12) gives the well known "string-functions" of $SU(2)$ is a non-trivial task.

* The indices k, k', $k+k'$ are understood modulo N.

The generalization to Wess–Zumino models with arbitrary group G and level N is now obvious. One tensors N copies of the G theory at level 1, which can be constructed a la Frenkel–Kac, out of free bosons, and then take the diagonal sum. In the non-simply laced case one has to bosonize the Ising fermion which is needed at level 1 using the result of [2]. In all of these cases one considers free bosons compactified on tori which are tensor products of maximum tori of simple Lie groups. One can then use this representation of current algebras to construct an arbitrary G/H model. This procedure is straightforward and we will not deal with it further.

One can not imagine the most general situation. Consider an arbitrary CFT which contains also $N \geq 1$ (2,0) operators in its spectrum. These operators, $\Phi_i(z)$, have to be primary fields. One can then write down the most general OPE consistent with conformal invariance and associativity of the operator algebra⋆

$$T(z)T(w) = \frac{1}{2}\frac{c}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + \cdots \tag{13a}$$

$$T(z)\Phi_i(w) = 2\frac{\Phi_i(w)}{(z-w)^2} + \frac{\partial_w \Phi_i(w)}{(z-w)} + \cdots \tag{13b}$$

$$\begin{aligned}\Phi_i(z)\Phi_j(w) = {} & \frac{\delta^{ij}}{(z-w)^4} + D^\alpha_{ij}\frac{J^\alpha(w)}{(z-w)^3} + \frac{4}{c}\delta^{ij}\frac{T(w)}{(z-w)^2} + C^k_{ij}\frac{\Phi_k(w)}{(z-w)^2} \\ & + \frac{1}{2}D^\alpha_{ij}\frac{\partial_w J^\alpha(w)}{(z-w)^2} + \frac{2}{c}\delta^{ij}\frac{\partial_w T(w)}{(z-w)} + \frac{1}{2}C^k_{ij}\frac{\partial_w \Phi_k(w)}{(z-w)} \\ & + \frac{1}{6}D^\alpha_{ij}\frac{\partial_w^2 J^\alpha(w)}{(z-w)} + F^k_{ij}\frac{W_k(w)}{(z-w)} + \cdots\end{aligned} \tag{13c}$$

where $J^\alpha(z)$ are possible (1,0) operators and $W_i(z)$ are possible (3,0) operators. Associativity constrains the OPE coefficients to have the following symmetry properties

$$D^\alpha_{ji} = -D^\alpha_{ij}, \quad F^k_{ji} = F^k_{ij}, \quad C^k_{ji} = C^k_{ij} \tag{14}$$

⋆ Such a line of thought has been also pursued by J. Harvey, with a slightly different goal: to decompose a CFT in its "irreducible" components.

In particular C_{ij}^k is completely symmetric in all indices. A general (2,0) operator has the form:

$$\tilde{T}(z) = \alpha_0 T(z) + \sum_{i=1}^{N} \alpha_i \Phi_i(z). \tag{15}$$

If one asks, what is the condition that $\tilde{T}(z)$ is a stress energy tensor, then the answer is easy. The α_0, α_i have to satisfy a system of quadratic equations:

$$\sum_{i=1}^{N} (\alpha_i)^2 = \frac{1}{2}\alpha_0(1-\alpha_0)c$$

$$\sum_{i=1}^{N} C_{ij}^k \alpha_i \alpha_j = 2(1-2\alpha_0)\alpha_k \quad k = 1, 2, 3, \ldots, N \tag{16}$$

whereas the new central charge $\tilde{c}$ for $\tilde{T}(z)$ is given by $\tilde{c} = \alpha_0 c$. If α_0, α_i is a solution of (16) then one can verify that $(1-\alpha_0), -\alpha_1$ is again a solution. Thus the theory with central charge c can be written as a product† of two theories with central charges $\alpha_0 c$ and $(1-\alpha_0)c$. In general of course there are more than two solutions to the system (16). It is also not obvious that the new central charge will come out rational. It is in fact true that for the constructions we described in the Lie algebra case that all solutions to (16) are rational.

A special case would be to consider (2,0) operators which are vertexs operators corresponding to vectors on a lattice with $(\text{length})^2 = 4$. Thus the general setting would be to consider a lattice and a set of vectors $\vec{\alpha}_i$, $i = 1, \ldots, 2N$ with the following properties: (1) if $\vec{\alpha}$ belongs to the set so does $-\vec{\alpha}$; (2) $\vec{\alpha} \cdot \vec{\alpha} = 4$; (3) if $\vec{\alpha}_1 \cdot \vec{\alpha}_2 = \pm 2$ then $\vec{\alpha}_1 \mp \vec{\alpha}_2$ belongs to the set; (4) if $\vec{\alpha}_1 \neq \pm\vec{\alpha}_2$ then $\vec{\alpha}_1 \cdot \vec{\alpha}_2 = \pm 3, \pm 2, \pm 1, 0$. If condition (1)–(4) are satisfied then one can construct new stress-energy tensors.‡ A classification of such systems of vectors can proceed along the lines of the Cartan classification.

There are lattices other than those of Lie Groups that satisfy the criteria above. A celebrated example is the 24-dimensional Leech-lattice where there exist an amazing total of 196,884 (2,0) operators. It is quite possible that by looking at lattices other than Lie group lattices, we can construct a rational CFT which does not belong to the list available today.

† By product here we mean a generalized product, not that the full partition function is a product.
‡ These results, as I learned recently have been found also by the authors fo ref. [3]. The only difference is that they overlooked the possible existence of (1,0) operators in the operator algebra of (2,0)operators.

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I recently learned from G. Dunne that he and collaborators were pursuing similar ideas, (see ref. [3]). J. Harvey has also been working in this direction.

REFERENCES

1. P. Goddard, A. Kent, D. Olive, Phys. Lett. **152B** (1985) 88. M. R. Douglas, Caltech preprint CALT-68-1453. F. A. Bais, P. Bouwknegt, M. Surridge, K. Schoutens, Nucl. Phys. **B304** (1988) 348.

   D. Kastor, E. Martinec, Z. Qiu, Chicago preprint EFI-87-58. J. bagger, D. Nemeschansky, S. Kankielowicz, Harvard preprint HUTP-87/A073.

2. E. Kiritsis, Phys. Lett. **198B** (1987) 379; Erratum, *ibid.* **199B** (1987) 597.

3. G. V. Dunne, I. G. Halliday, P. Suranyi, preprint Imperial /TP/87-88/25.