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A Bosonic Representation of the Ising Model

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Abstract

A bosonic construction of the critical Ising Model is given. The stress-energy tensor has a different form from that of a free scalar field. Equivalence is established on an arbitrary compact Riemann surface. It is argued that this description is realized in a sector of the critical Askin-Teller model.

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Recently there has been a lot of progress in understanding critical phenomena in two dimensions, [1,2]. The specific reasons for that are, the fact that the two-dimensional conformal group is infinite dimensional, providing severe constraints on possible conformal field theories and the recent surge of interest for string theories which heavily rely on two-dimensional conformal field theory. There is a serious hope that all universality classes of two-dimensional critical phenomena will be described algebraically through the conformal group and its extensions. Another special feature of two dimensions is bosonization, that is, the equivalence, in a certain sense, between a bosonic and fermionic theory.

The purpose of this letter is to present a certain "bosonization" of the critical Ising model. One might ask, why is it worth doing it after all, in particular for a model which is the simplest possible one? There are two reasons for this. First the bosonization prescription, as we discuss below, explains the existence of critical points in the Askin-Teller model, belonging to the Ising universality class. Second, we shall show that the equivalence of the two models is true on an arbitrary compact Riemann surface, a fact that might prove useful in teaching us how to handle conformal field theories on a Riemann surface.

One might wonder how can, a free scalar field with $c = 1$ (c is the conformal anomaly), be equivalent to an Ising fermion ($c = \frac{1}{2}$)? The answer is that the stress-energy tensor of the bosonic model does not have the standard quadratic form. In particular, as we shall see, it is far from obvious that the scalar field is free.

The construction that we are going to describe is at the operator level. Consider a scalar field $\phi(z,\bar{z})$ taking values in a circle of radius R, (for our purposes, $R = 1$). We will assume that the two point function is the free one:†

$$\langle 0|\phi(z)\phi(w)|0\rangle = -\ln(z-w). \tag{1}$$

The operators in this theory are the so-called Vertex operators, $V_a(z) \equiv : e^{ia\phi(z)} :$, and their derivatives.

The holomorphic part of the stress-energy tensor, from general arguments, must satisfy the following operator product expansion,‡ (O.P.E.)

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)}. \tag{2}$$

† From now on we suppress the $\bar{z}$ dependence, which can be substituted in the end.

‡ We supress regular pieces in the O.P.E.

In particular it has (holomorphic) scaling dimension two.

Consider the most general operator of dimension two. It is a linear combination of $:\partial_z\phi\partial_z\phi:$ $, \partial_z^2\phi, V_{\pm 2}(z), \partial_z\phi V_{\pm\sqrt{2}}(z)$. Thus we will consider a general linear combination of the operators above. If we impose (2), then there are two distinct possibilities: The first is $T(z) = -\frac{1}{2} : \partial_z\phi\partial_z\phi : +\beta\partial_z^2\phi$, which has been known already from the work of ref. [3]. The second is:

$$T(z) = -\frac{1}{4} : \partial_z\phi\partial_z\phi : +\beta V_2(z) + \bar{\beta} V_{-2}(z). \tag{3}$$

with $\beta\bar{\beta} = \frac{1}{16}$, and $\beta, \bar{\beta}$ are otherwise arbitrary. In this case a direct computation shows that $c = \frac{1}{2}$! ¿From now on we will focus on the second case.

The value of the central charge hints that somehow the theory described by (3) is the Ising model. Let's investigate, what are the primary operators in this theory.

Recall that a primary operator $\Phi(z)$, of dimension Δ, satisfies the following O.P.E.

$$T(z)\Phi(w) = \Delta\frac{\Phi(w)}{(z-w)^2} + \frac{\partial_w\Phi(w)}{(z-w)} \tag{4}$$

It is easy to show that derivatives of ϕ cannot be primary operators. But what about vertex operators? Since $V_a \otimes V_b \simeq V_{a+b}$, only $V_{\pm 1}$ have a chance of being primary. In fact, by imposing (4), we can deduce that $\psi(z) = kV_1(z) + \bar{k}V_{-1}(z)$ is primary if and only if $4\beta\bar{k} = k$, and its dimension is $\Delta = \frac{1}{2}$. The dimension suggests that this operator represents the fermion of the Ising model. There is another operator that we have to look for, the spin field (order and disorder operator), with $\Delta = \frac{1}{16}$. In the standard free scalar theory there is an operator, (in fact two, $H^{\pm}(z)$), of dimension $\frac{1}{16}$, the "twist fields" of the boson, [4]. In the presence of the twist fields the boson is antiperiodic (the twist fields are creating cuts on the complex plane, for ϕ. We need though to compute again the dimension of these operators using the new form of the energy momentum tensor, (3). A straightforward calculation gives, $\Delta_H = \frac{1}{32} + \frac{\beta+\bar{\beta}}{16}$. Thus in order for $H^{\pm}$ to have the correct dimension, $\beta + \bar{\beta} = \frac{1}{2}$, which fixes them completely, $\beta = \bar{\beta} = \frac{1}{4}$.

Thus:

$$T(z) = -\frac{1}{4} : \partial_z\phi\partial_z\phi : +\frac{1}{4}(V_2(z) + V_{-2}(z)) \equiv -\frac{1}{4} : \partial_z\phi\partial_z\phi : +\frac{1}{2} : \cos(2\phi) : \tag{5a}$$

$$\Psi(z) = \frac{1}{\sqrt{2}}(V_1(z) + V_{-1}(z)) \equiv \sqrt{2} : \cos\phi :, \langle 0|\psi(z)\psi(w)|0\rangle = \frac{1}{z-w} \tag{5b}$$

The next step is to verify the operator algebra of the Ising model:

$$\left[\frac{1}{2}\right]\otimes\left[\frac{1}{2}\right]=[0],\ \left[\frac{1}{2}\right]\otimes\left[\frac{1}{16}\right]=\left[\frac{1}{16}\right],\ \left[\frac{1}{16}\right]\otimes\left[\frac{1}{16}\right]=[0]\oplus\left[\frac{1}{2}\right] \tag{6}$$

In order to compute expectation values of vertex operators in the presence of twist fields we have to be careful about the winding number of the boson. We will not go into detail here, but we will just mention that bosonic operators with non-zero winding number, interpolate in general between the two distinct twist fields $H^+(z)$ and $H^-(z)$,[5]. The action of such operators on the twist fields $H^\pm$ is given by a 2×2 matrix. A simple calculation, [5], shows that this matrix, S, for $V_{\pm 1}(z)$, is, $S=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$, so that:

$$\langle 0|H^+(z_1)\Psi(z_2)H^+(z_3)|0\rangle=\langle 0|H^-(z_1)\Psi(z_2)H^-(z_3)|0\rangle=0 \tag{7a}$$

$$\langle 0|H^+(z_1)\Psi(z_2)H^-(z_3)\rangle=\frac{1}{\sqrt{2}}z_{13}^{-\frac{1}{8}}\left[\frac{z_{13}}{z_{12}z_{23}}\right]^{\frac{1}{2}} \tag{7b}$$

Thus (7a,b) qualitatively verifies (6), under the identifications, $\sigma\equiv H^+$, $\mu\equiv H^-$.

The dihedral symmetry of the bosonic system translates into the $\mathbf{Z}_2$ symmetry of the Ising model and its dual $\widetilde{\mathbf{Z}}_2$.

Next we calculate the 4-point functions in the bosonic theory. The following two are very simple to calculate using:

$$\left\langle 0\left|\prod_{i=1}^{n}:e^{ia_i\phi(z_i)}\right|0\right\rangle=\prod_{i<j}^{n}(z_{ij})^{a_ia_i} \tag{8}$$

$$\langle 0|\Psi(z_1)\psi(z_2)\Psi(z_3)\Psi(z_4)|0\rangle=\frac{1}{z_{14}z_{23}}\left[\frac{x^2-x+1}{x}\right],x=\frac{z_{12}z_{34}}{z_{13}z_{24}} \tag{9a}$$

$$\frac{\langle 0|\Psi(z_1)\Psi(z_2)H^\pm(z_3)H^\pm(z_4)|0\rangle}{\langle 0|H^\pm(z_3)H^\pm(z_4)|0\rangle}=\frac{1}{2}\frac{z_{34}}{z_{14}z_{23}}\frac{x-2}{x}\sqrt{1-x} \tag{9b}$$

The correlation function of four twist fields is more difficult to calculate. Fortunately, in ref. [6], such a correlation function has been calculated using coverings of the sphere. We can employ this calculation with minor modifications to show that, for example:

$$
\begin{aligned}
&\langle 0|H^{\pm}(z_1,\bar z_1)H^{\pm}(z_2,\bar z_2)H^{\pm}(z_3,\bar z_3)H^{\pm}(z_4,\bar z_4)|0\rangle \\
&\qquad =\frac{1}{2}[z_{12}\bar z_{12}z_{34}\bar z_{34}]^{-\frac{1}{8}}[x\bar x(1-x)(1-\bar x)]^{-\frac{1}{8}}G(x,\bar x)
\end{aligned}
\tag{10}
$$

$$
G(x,\bar x)=\sqrt{1-\sqrt{x}}\sqrt{1-\sqrt{\bar x}}+\sqrt{1+\sqrt{x}}\sqrt{1+\sqrt{\bar x}}
$$

(9a,b) and (10) coincide with the correlation function of the Ising model.

As a final check we complete the partition function of the bosonic theory on a strip with periodic boundary conditions (that is, on a torus).

The method relies on computing $\langle T\rangle$ and integrating with respect to the modulus of the torus, τ, to obtain the partition function.

In order to compute $\langle T\rangle$ in the bosonic theory we need the propagator for the scalar field on the torus. We will employ the results on chiral bosonization, [7]. The path integral over the torus contains also a sum over the instanton sectors. Thus we split the scalar field ϕ into a classical, (instanton) part and a quantum part, $\phi=\phi_{cl}+\phi_{qu}$. Then

$$
\langle 0|\phi_{qu}(z)\phi_{qu}(w)|0\rangle = -\ln E(z,w),\; E(z,w)=\frac{\theta_1(z,w|\tau)}{\theta_1'(0|\tau)}, \tag{11}
$$

where θ_1 is the standard θ-function on the torus, [8].

$$
\langle T(z)\rangle = -\frac{1}{4}\lim_{w\to z}\left\{\langle\partial_z\phi(z)\partial_w\phi(w)\rangle+\frac{1}{(z-w)^2}\right\} \tag{12}
$$

since the expectation value of $V_{\pm 2}(z)$ vanishes. A straightforward computation gives $\langle T(z)\rangle=\frac{e_\nu}{4}$, where

$$
e_\nu=-4\pi i\frac{\partial}{\partial\tau}\ln\left[\frac{\theta_{\nu+1}(0|\tau)}{\eta(\tau)}\right],\; \nu=1,2,3 \tag{13}
$$

and ν labels the periodicity properties of the fermion operator and $\eta(\tau)$ is the Dedekind η-function. (In the bosonic theory, this is generated by an appropriate sum over instanton sectors, [7].) $\nu=1,2,3$ corresponds to (P,AP), (AP,AP), (AP,P) boundary conditions.

Integrating with respect to τ we obtain

$$\mathbf{Z}_\nu \propto \left[\frac{\theta_{\nu+1}(0|\tau)}{\eta(\tau)}\right]^{\frac{1}{2}} \tag{14}$$

Thus the partition function of the bosonic theory is given by the sum over the various sectors,

$$\mathbf{Z}_{tot} = \sum_{\nu=1}^{3} \mathbf{Z}_\nu(\tau)\bar{\mathbf{Z}}_\nu(\bar{\tau}) \tag{15}$$

which is equal to the partition function of the Ising model:

$$\mathbf{Z}_{Ising} = |\chi_0|^2 + |\chi_{\frac{1}{2}}|^2 + |\chi_{\frac{1}{16}}|^2 \tag{16}$$

where χ_0, $\chi_{\frac{1}{2}}$, $\chi_{\frac{1}{16}}$ are the appropriate characters of the Virasoro algebra for $c = \frac{1}{2}$.

What we have shown so far is that the bosonic theory defined by (1) and (5) is equivalent to the Ising model on the sphere and the torus. In fact, we can do more. We will now show that the two theories possess the same partition function on any compact Riemann surface. To achieve that we will show that the expectation value of the stress-energy tensor is the same in both theories and thus their partition function are the same up to trivial constant.⋆

Let's first compute $\langle T \rangle$ in the fermionic case. The two-point function of the fermion on a compact Riemann surface of genus $g \geq 2$ is given by the Szego kernel†

$$\langle 0|\Psi(z)\Psi(w)|0\rangle = \frac{\theta\left[{a \atop b}\right]\left(\int_w^z \nu\right)}{\theta\left[{a \atop b}\right](0)} \cdot \frac{1}{E(z,w)} \equiv P\begin{bmatrix} a \\ b \end{bmatrix}(z,w) \tag{17}$$

where the pair (a, b), (a, b are g-dimensional vectors whose components are either 0 or $\frac{1}{2}$), specifies an arbitrary even spin-structure on the surface, $E(z, w)$ is the prime form, and ν^i,

⋆ We will only discuss even spin structures where there are no zero modes for the fermion.
† For notation and more details see ref. [9].

$i = 1, 2, ..., g$, is a basis of holomorphic one-forms.

$$\langle T(z)\rangle_F = -\frac{1}{2}\lim_{w\to z}\left\{\langle\Psi(z)\partial_w\Psi(w)\rangle - \frac{1}{(z-w)^2}\right\} \tag{18}$$

The Szego kernel satisfies the following identity, [9],

$$\left[P\begin{bmatrix}a\\b\end{bmatrix}(z,w)\right]^2 = \omega(z,w) + \sum_{i,j=1}^{g} A_{ij}\nu^i(z)\nu^j(w) \tag{19}$$

$$A_{ij} \equiv \frac{\partial^2 \ln\theta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]}{\partial z_i\partial z_j}[0],\ \omega(z,w) = \frac{\partial^2}{\partial z\partial w}\ln E(z,w). \tag{20}$$

We need also the short distance expansion of the prime form.

$$E(z,w) = (z-w) - \frac{(z-w)^3}{12}S(w) + O[(z-w)^5] \tag{21}$$

where S is the projective connection, [9]. Using (18), (19), (21) we can easily show that

$$\langle T(z)\rangle_F = \frac{1}{4}\sum_{i,j=1}^{g} A_{ij}\nu^i(z)\nu^j(z) - \frac{S(z)}{24} \tag{22}$$

Note that $\langle T(z)\rangle_F$ depends on z, since translation invariance is not a symmetry of the correlation functions for $g > 1$.

The corresponding calculation in the bosonic model proceeds along the same lines.

$$\langle T(z)\rangle_B = -\frac{1}{4}\lim_{w\to z}\left\{\langle\partial_z\phi(z)\partial_w\phi(w)\rangle + \frac{1}{(z-w)^2}\right\} \tag{23}$$

$$\phi(z) \equiv \sum_{i=1}^{g} p^i \int_{P_o}^{z} \nu^i + \phi_{qu}(z)$$

where the winding number p^i takes the appropriate values, $p^i = 2m + a^i$, $m \in \mathbf{Z}$, P_0 is an arbitrary point on the surface and

$$\langle\partial_z\phi_{qu}(z)\partial_w\phi_{qu}(w)\rangle = -\partial_z\partial_w lnE(z,w) = -\omega(z,w) \tag{24}$$

The sum over instanton sectors is weighed by the holomorphic instanton action, $S_m \equiv \frac{1}{2}(m + a)^i\Omega_{ij}(m+a)^j + 2\pi i b^i(m+a)^i$, where Ω_{ij} is the period matrix of the surface. The instanton sum contributes a factor $-2\frac{\partial}{\partial\Omega_{ij}}\ln\theta\left[{a \atop b}\right](0)\nu^i(z)\nu^i(z)$ which by the heat equation satisfied by the θ-functions is equal to $-\sum_{i,j=1}^{g} A_{ij}\nu^i(z)\nu^j(z)$. Thus $\langle T(z)\rangle_F = \langle T(z)\rangle_B$ which completes the proof.

Concerning the physical relevance of the previous bosonic construction of the critical Ising model we could ask the following question. Is there a critical statistical system belonging to the Ising universality class where the previous description is the natural one? The answer is yes! The relevant system is the Askin-Teller (A-T) model.

The A-T model is described by two Ising spin coupled with a four-spin interaction.⋆ There are two couplings, β, governing the strength of the four-spin interaction and λ, governing the spin-spin interaction. At $\beta = 1$ the strength of the four-spin interaction vanishes and there is a line of critical points, $-1 \leq \lambda \leq 1$, of the Kosterlitz-Thouless type, with continuously varying critical exponents. The point $\lambda = 0$, corresponds to two decoupled critical Ising models, whereas at $\lambda = \pm 1$ the model has $\mathbf{Z}_4$ symmetry, corresponding to the critical Potts model.

Starting at $\lambda = 1$, the critical line separates in two other critical lines, which are of the Ising type and separate a completely ordered phase, a partially ordered phase, and a disordered one. We claim that our construction is realized at these critical lines for $\lambda > 1$.

The Hamiltonian in this region, in fact, has the form of our stress-energy tensor, that is, it contains also a cosine term whose coefficient at the Ising transition point is a constant independent of λ, β while the coefficient of the quadratic term is linear in λ, [10]. We can easily see how this arises in our construction, using,

$$\langle 0|\phi(z)\phi(w)|0\rangle = -\frac{1}{\lambda}\ln(z-w) \tag{25}$$

since the action is linear in λ. The canonical stress-energy tensor becomes:

$$T(z) = -\frac{\lambda}{4} :\partial_z\phi\partial_z\phi: +\frac{1}{2} :\cos\frac{2\phi}{\sqrt{\lambda}}: \tag{26}$$

and exhibits the aforementioned properties. It is a trivial task to verify that the modification (25) does not alter any of the other relations we derived before.

⋆ For more details see ref. [10]. We use the notation and convention of the same reference.

This bosonic representation of a Majorana-Weyl fermion might also be useful, concerning aspects of conformal field theory on higher genus Riemann surfaces. In particular correlation functions seem to be easier to evaluate in this picture. It might hint also, on how to resolve some ambiguities concerning the handling of fermion zero modes on Riemann surfaces. Work in this direction is in progress.

To summarize, we gave a bosonic construction of the critical Ising model, using a free (?) scalar field, and a non-standard form of its stress-energy tensor. We showed the equivalence at the level of the operator algebra and the correlation functions. We gave also a proof that the partition functions of the two models are identical on any compact Riemann surface. We finally showed that this construction describes the two critical lines of the Ising universality class in the A-T model.

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