The Structure of N=2 Superconformally Invariant Unitary "Minimal" Theories: Operator Algebra and Correlation Functions[★]

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Abstract

N=2 superconformal invariant theories are studied and their general structure is analyzed. The geometry of N=2 complex superspace is developed as a tool to study the correlation functions of the theories above. The Ward Identities of the global N=2 superconformal symmetry are solved, to restrict the form of correlation functions. Advantage is taken of the existence of the degenerate operators to derive the "fusion" rules for the unitary minimal systems with $\tilde{c} < 1$. In particular, the closure of the operator algebra for such systems is shown. The $\tilde{c} = 1/3$ minimal system is analyzed and its two-, three-, and four-point functions as well as its operator algebra are calculated explicitly.

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1. Introduction

Recently there has been a lot of progress in understanding 2-d critical phenomena from the quantum field theory point of view, [1]

A statistical system at the critical point is scale invariant and, in two dimensions, it has been proven that, under mild assumptions, it is invariant under the full conformal group which is infinite dimensional, [2]. B.P.Z., in ref. [1], laid the foundation of 2-d conformal field theory. Their techniques have been proven very powerful in analyzing the structure or conformal field theories, and in answering questions like, what is the representation content of a given theory, what is its operator algebra, how can one calculate exactly correlation functions, and so on.

Conformal theory in two dimensions attracted also a lot of interest because it gives a nice natural framework in formulating (super)string theories, which are supposed to be candidates for a unified theory of nature, [3].

Conformal invariance puts severe constraints on the structure of a theory. By restricting the unitary representations of the conformal group, one can classify all possible anomalous dimensions that can appear in two-dimensional critical systems. In particular, for $c < 1$, (c is the anomaly of the conformal algebra), there exists only a discrete list of values of c corresponding to theories which are unitary, (i.e., theories that do not have negative norm states in their Hilbert space), [4]. The first few of these theories have been identified as well known statistical systems, without continuous symmetries, the Ising model, ($c = 1/2$), the $\mathbf{Z}_3$ Potts model, ($c = 4/5$), the tricritical Ising model, ($c = 7/10$), and the tricritical $\mathbf{Z}_3$ Potts model ($c = 6/4$). Knowledge of the structure of these theories allows one to better understand the universality classes of critical behavior.

This subset of conformal two-dimensional field theories contain some special representations, (the so-called degenerate representations), the presence of which renders the theory exactly solvable, in the sense that all correlation functions satisfy linear differential equations, so that, in the worst case, it is up to a reasonable computer to evaluate them.

A natural extension of conformal invariance, inspired by attempts to construct fermionic string theories, has been its extension to include supersymmetry. Supersymmetry, so far, although having appealing theoretical advantages, has been elusive , despite a lot of both theoretical and experimental efforts to give some clues supporting its existence.

It was surprising that the first example of supersymmetry observed in nature, came from a two-dimensional critical system, the tricritical Ising model, [5], realized experimentally by adsorbing helium on krypton-plated graphite, [6].

Superconformal invariance proves to be very fruitful also in the formulation of (super)string theories, where superconformal techniques are indeed valuable, making a lot of problems, which otherwise are difficult to tackle, almost elementary, [3, 7, 8].

A natural extension of N=1 superconformal invariance is to include an extended supersymmetry, giving thus rise to N=2 superconformal invariance. There has been considerable activity recently on this subject, [24,9,10,11,12]. The Kač determinant has been calculated by several authors and it was used to study the irreducible unitary representations of the N=2 superconformal algebra, [9, 11]. The characters of the algebra have also been calculated, [12].

One might wonder about the utility of an extended superconformal algebra. There are various reasons explaining why there is such an interest.

First, the N=2 superconformal algebra is the gauge algebra of the U(1) string, [13], which, despite its phenomenological insignificance, is a good toy model to try techniques pertinent in string theories, both first and second quantized. From the point of view of superstring theories there is a need for N=2 superconformal invariance in the associated non-linear σ-model on a compact Ricci flat manifold, to ensure N=1 supersymmetry in four dimensions, [14]. We need supersymmetry to be exact after compactification and non-perturbatively stable. There is a serious hope that N=2 superconformal techniques will provide useful tools in disentangling various questions pertaining to the nature of different possible compactifications.

Another reason is that N=2 invariance might be relevant in some critical statistical systems.

Such an example has been given in [15], where it was shown that the O(2) Gaussian model (XY model) is N=2 superconformal invariant for a specific value of the radius, see also [26].

In this paper we pursue our aim toward understanding the N=2 superconformal field theory. We develop some tools for doing complex analysis in N=2 superspace, and we use them to solve the Ward identities for the global N=2 symmetry, and find the general structure of the correlation functions. The two- and three-point functions in particular, are almost fixed. We derive the "fusion" rules for the degenerate representations of the N=2 algebra with $\tilde{c} < 1$, and we prove the closure of the degenerate representation content of the unitary "minimal" theories. The simplest minimal system, with $\tilde{c} = 1/3$, is analyzed in detail. We calculate the four-point function and we derive its operator algebra.

The structure of this paper is the following: In section 2 we introduce N=2 supersymmetry and we develop the local analytic geometry of N=2 complex superspace. We characterize the global N=2 superconformal group $OSP(2|2)$ and we give the explicit form of a general global supertransformation. In section 3 we analyze the general structure of N=2 superconformal invariant theories. Section 4 is devoted to the analysis of the structure of the correlation functions, implied by global N=2 superconformal invariance, and in the derivation of "fusion" rules for the operator product expansion of the minimal theories with $\tilde{c} < 1$. In section 5 the simplest N=2 superconformal invariant system with $\tilde{c} = 1/3$ is analyzed. We compute its operator algebra and the four-point function. Section 6 contains our conclusions. In Appendix A we use the Feigin-Fuks construction to derive the two-point functions in the R and T sectors of an N=2 superconformal system. In Appendix B we derive and solve, up to level 5/2, the superdifferential equations satisfied by the three-point function, containing degenerate operators. In Appendix C we give a justification of the "fusion" rules, through the construction of the degenerate representations with $\tilde{c} < 1$ using free fermions. In appendix D we construct the primary fields of the $\tilde{c} = 1/3$ theory as vertex operators in the Gaussian model, and we explicitly check some of the results of section 5.

2. N=2 Supersymmetry and the Analytic Geometry of N=2 Complex Superspace

N=2 supersymmetry is a natural extension of N=1 supersymmetry. In this case we have two different supersymmetry generators (supercharges), as well as an O(2) (or U(1)) current which manifests the symmetry of the theory under an O(2) rotation of the two supersymmetries. The natural space to define the fields of the theory is N=2 superspace, (or more precisely (2,0) superspace). In a theory with (super)conformal invariance the left and right sectors of the theory completely decouple, so that the structure of the theory is that of a tensor product of the left and right sectors. From now on we will restrict ourselves to the left sector only, keeping in mind the previous remarks.

(2,0) superspace includes, apart from the complex analytic coordinate z, two other fermionic coordinates, θ and $\bar{\theta}$ corresponding to the two supersymmetries.

$$\theta^2 = \bar{\theta}^2 = \{\theta, \bar{\theta}\} = 0. \tag{2.1}$$

A point in superspace will be denoted by $\mathbf{z} \equiv (z, \theta, \bar{\theta})$.

A superfield is an analytic function in $\mathbf{z}$ defined through its power series expansion in the fermionic coordinates:

$$\Phi(\mathbf{z}) \equiv \phi(z) + \theta\bar{\psi}(z) + \bar{\theta}\psi(z) + \theta\bar{\theta}g(z). \tag{2.2}$$

The two supersymmetry transformations can be written as:

$$(z, \theta, \bar{\theta}) \rightarrow (z - \epsilon\bar{\theta}, \theta + \epsilon, \bar{\theta}) \tag{2.3a}$$

$$(z, \theta, \bar{\theta}) \rightarrow (z - \bar{\epsilon}\theta, \theta, \bar{\theta} + \bar{\epsilon}) \tag{2.3b}$$

where $\epsilon, \bar{\epsilon}$ are anticommuting variables which are the parameter of the transformation. Under

the two supersymmetry transformations, (2.3a,b), a superfield transforms as:

$$\Phi(z,\theta,\bar{\theta}) \rightarrow \Phi(z - \epsilon\bar{\theta}, \theta + \epsilon, \bar{\theta}) = \phi(z - \epsilon\bar{\theta}) + (\theta + \epsilon)\bar{\psi}(z - \epsilon\bar{\theta}) + \bar{\theta}\psi(z - \epsilon\bar{\theta})$$

$$+(\theta + \epsilon)\bar{\theta}g(z - \epsilon\bar{\theta}) = \phi(z) + \epsilon\bar{\psi}(z) + \bar{\theta}[\epsilon\partial_z\phi(z) - \epsilon g(z) + \psi(z)]$$

$$+\theta\bar{\psi}(z) + \theta\bar{\theta}[g(z) + \epsilon\partial_z\bar{\psi}(z)] \tag{2.4a}$$

$$\Phi(z,\theta,\bar{\theta}) \rightarrow \Phi(z - \bar{\epsilon}\theta, \theta, \bar{\theta} + \epsilon) = \phi(z - \bar{\epsilon}\theta) + \theta\bar{\psi}(z - \bar{\epsilon}\theta) + (\bar{\theta} + \bar{\epsilon})\psi(z - \bar{\epsilon}\theta)$$

$$+\theta(\bar{\theta} + \bar{\epsilon})g(z - \bar{\epsilon}\theta) = \phi(z) + \bar{\epsilon}\psi(z) + \theta[\bar{\psi}(z) + \bar{\epsilon}\partial_z\phi(z) + \bar{\epsilon}g(z)]$$

$$+\bar{\theta}\psi(z) + \theta\bar{\theta}[g(z) - \bar{\epsilon}\partial_z\psi(z)] \tag{2.4b}$$

which implies the following transformation laws for the component fields:

$$
\begin{aligned}
\delta_\epsilon\phi(z) &= \epsilon\bar{\psi}(z) & \bar{\delta}_{\bar{\epsilon}}\phi(z) &= \bar{\epsilon}\psi(z) \\
\delta_\epsilon\psi(z) &= \epsilon[\partial_z\phi(z) - g(z)] & \bar{\delta}_{\bar{\epsilon}}\psi(z) &= 0 \\
\delta_\epsilon\bar{\psi}(z) &= 0 & \bar{\delta}_{\bar{\epsilon}}\bar{\psi}(z) &= \bar{\epsilon}[\partial_z\phi(z) + g(z)] \\
\delta_\epsilon g(z) &= \epsilon\partial_z\bar{\psi}(z) & \bar{\delta}_{\bar{\epsilon}}g(z) &= -\bar{\epsilon}\partial_z\psi(z)
\end{aligned}
\tag{2.5}
$$

It is easy to verify the global supersymmetry algebra:

$$[\delta_\epsilon, \delta_{\bar{\epsilon}}] = 2\epsilon\bar{\epsilon}\frac{\partial}{\partial z}, \qquad [\delta_\epsilon, \delta_\epsilon] = [\delta_{\bar{\epsilon}}, \delta_{\bar{\epsilon}}] = 0. \tag{2.6}$$

The covariant derivatives in superspace are defined by:

$$D \equiv \frac{\partial}{\partial\theta} + \bar{\theta}\frac{\partial}{\partial z} \quad , \qquad \bar{D} \equiv \frac{\partial}{\partial\bar{\theta}} + \theta\frac{\partial}{\partial z} \tag{2.7a}$$

$$D^2 = \bar{D}^2 = 0 \quad , \qquad \{D, \bar{D}\} = 2\frac{\partial}{\partial z}. \tag{2.7b}$$

We introduce here the notion of a chiral N=2 superfield, as a superfield satisfying one of the

following conditions:

$$D\Phi(\mathbf{z}) = 0 \implies \Phi(\mathbf{z}) = \phi(z) + 2\bar{\theta}\psi(z) - \theta\bar{\theta}\partial_z\phi(z) \tag{2.8a}$$

$$\bar{D}\bar{\Phi}(\mathbf{z}) = 0 \implies \Phi(\mathbf{z}) = \bar{\phi}(z) + 2\theta\bar{\psi}(z) + \theta\bar{\theta}\partial_z\bar{\phi}(z) \tag{2.8b}$$

We can write an action for a free chiral superfield, (its left part), as:

$$S \equiv \frac{i}{8\pi}\int dz d\bar{z} d\theta d\bar{\theta}[\Phi(\mathbf{z},\bar{z})\partial_{\bar{z}}\Phi(\mathbf{z},\bar{z}) - \bar{\Phi}(\mathbf{z},\bar{z})\partial_{\bar{z}}\phi(\mathbf{z},\bar{z})]. \tag{2.9}$$

The Grassman integration is defined through the usual standard rules:

$$\int d\theta d\bar{\theta} = \int d\theta d\bar{\theta}\theta = \int d\theta d\bar{\theta}\bar{\theta} = 0 \quad , \quad \int d\theta d\bar{\theta}\,\bar{\theta}\theta = 1. \tag{2.10}$$

If we define $\phi_{1,2} \equiv 1/2(\phi \pm i\bar{\phi})$, the action in component form is:

$$S = \frac{1}{2\pi}\int dz d\bar{z}[\partial_z\phi_1\partial_{\bar{z}}\phi_1 + \partial_z\phi_2\partial_{\bar{z}}\phi_2 + i\psi\partial_{\bar{z}}\bar{\psi} + i\bar{\psi}\partial_z\psi] \tag{2.11}$$

showing explicitly the on-shell degrees of freedom, that is two self-dual scalar fields and two Majorana-Weyl fermions.

If we call the generators of the two supersymmetries $G_{-1/2}, \bar{G}_{-1/2}$ then eq. (2.6) is translated into:

$$\{G_{-1/2}, G_{-1/2}\} = \{\bar{G}_{-1/2}, \bar{G}_{-1/2}\} = 0, \qquad \{G_{-1/2}, \bar{G}_{-1/2}\} = 2L_{-1} \tag{2.12}$$

L_{-1} being the usual translation operator on the complex plane. The full superconformal symmetry is generated by the usual Virasoro generators L_n, the supersymmetry generators,

$$G_r \equiv \frac{2}{n+1}[L_n, G_{-1/2}],\ \bar{G}_r \equiv \frac{2}{n+1}[L_n, \bar{G}_{-1/2}],\ r = n - 1/2 \tag{2.13}$$

and the U(1) current generators, J_n, which implement the U(1) symmetry, under which the two supercurrents are in complex conjugate representations. The full N=2 superconformal algebra

then takes the form:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{\tilde{c}}{4}(m^3 - m)\delta_{m+n,0}$$

$$[L_m, G_r] = (\frac{m}{2} - r)G_{m+r} \quad , \quad [L_m, \bar{G}_r] = (\frac{m}{2} - r)\bar{G}_{m+r}$$

$$[J_m, J_n] = \tilde{c}m\,\delta_{m+n,0} \quad , \quad [J_m, G_r] = G_{m+r}, [J_m, \bar{G}_r] = \bar{G}_{m+r} \tag{2.14}$$

$$\{G_r, G_s\} = \{\bar{G}_r, \bar{G}_s\} = 0 \quad , \quad [L_m, J_n] = -nJ_{m+n}$$

$$\{G_r, \bar{G}_s\} = 2L_{r+s} + (r-s)J_{r+s} + \tilde{c}(r^2 - \frac{1}{4})\delta_{r+s,0}$$

It is the generating algebra of N=2 superanalytic transformations in N=2 superspace. We should, at this point, define what we mean by an extended superanalytic transformation. The most general coordinate transformation in N = 2 superspace has the form[⋆]

$$z' = f_0(z) + \theta f_1(z) + \bar{\theta}\bar{f}_1(z) + \theta\bar{\theta}f_2(z)$$

$$\theta' = g_0(z) + \theta g_1(z) + \bar{\theta}\bar{g}_1(z) + \theta\bar{\theta}g_2(z) \tag{2.15}$$

$$\bar{\theta}' = h_0(z) + \theta h_1(z) + \bar{\theta}\bar{h}_1(z) + \theta\bar{\theta}h_2(z)$$

A natural definition for an extended superanalytic transformation is one under which the covariant derivatives transform homogeneously. Under (2.15) the covariant derivatives transform as:

$$D = (D\theta')D' + (D\bar{\theta}')\frac{\partial}{\partial\bar{\theta}'} + [Dz' - (D\theta')\bar{\theta}']\frac{\partial}{\partial z'} \tag{2.16a}$$

$$\bar{D} = (\bar{D}\bar{\theta}')\bar{D}' + (\bar{D}\theta')\frac{\partial}{\partial\theta'} + [\bar{D}z' - (\bar{D}\bar{\theta}')\theta']\frac{\partial}{\partial z'}. \tag{2.16b}$$

Consequently the conditions for (2.15) to be a superanalytic transformation are:[†]

$$\bar{D}\theta' = D\bar{\theta}' = Dz' - (D\theta')\bar{\theta}' = \bar{D}z' - (\bar{D}\bar{\theta}')\theta' = 0. \tag{2.17}$$

[⋆] $f_0, f_2, g_1, \bar{g}_1, h_1, \bar{h}_1$ are commuting functions, whereas $f_1, \bar{f}_1, g_0, g_2, h_0, h_2$ are anticommuting ones.

[†] In fact, even if we demand that D transforms in general as $D = (D\theta')D + (D\bar{\theta}')\bar{D}'$ we end up at (2.17). There is a dual requirement, $D = (D\bar{\theta}')\bar{D}'$ which gives conditions conjugate to (2.17)

Solving (2.17) we arrive at the most general form of an extended superanalytic transformation:

$$z' = f_0(z) + \theta g_1(z)h_0(z) + \bar{\theta}\bar{h}_1(z)g_0(z) + \theta\bar{\theta}[g_0(z)h_0(z)]'$$

$$\theta' = g_0(z) + \theta g_1(z) + \theta\bar{\theta}g_0'(z) \tag{2.18}$$

$$\bar{\theta}' = h_0(z) + \bar{\theta}h_1(z) - \theta\bar{\theta}h_0'(z)$$

along with the supplementary condition:

$$f_0'(z) = g_0'(z)h_0(z) - g_0(z)h_0'(z) + g_1(z)\bar{h}_1(z) \tag{2.19}$$

where in (2.19) and in the left-hand side of (2.18) a prime means differentiation with respect to z.

In particular the global supersymmetry transformations are special cases of (2.18) with $f_0(z) = z, g_0(z) = \epsilon, h_0(z) = 0, g_1(z) = \bar{h}_1(z) = 1$ and $f_0(z) = z, g_0(z) = 0, h_0(z) = \bar{\epsilon}, g_1(z) = \bar{h}_1(z) = 1$ respectively.

We define the two abelian N=2 superdifferentials through their transformation properties under analytic superconformal transformations:

$$d\mathbf{z}' \equiv (D\theta')d\mathbf{z} \quad , \quad d\bar{\mathbf{z}}' \equiv (\bar{D}\bar{\theta}')d\bar{\mathbf{z}} \tag{2.20}$$

(the bar over $d\bar{\mathbf{z}}$ should not be confused with the antiholomorphic coordinate $\bar{z}$, it denotes an independent abelian differential).

The superconformal tensor fields are defined by the condition that

$$\Phi(\mathbf{z})(d\mathbf{z})^{\Delta+Q/2}(d\bar{\mathbf{z}})^{\Delta-Q/2}$$

is an N=2 superconformal invariant quantity, where Δ, Q are the dimensions and charge of the lowest component field. They are the primary superfields generating the highest weight irreducible representations of the N=2 superconformal algebra. Globally defined tensor superfields must have dimensions and charges which are integers or half integers. They can be constructed as composite operations from locally defined fields.

We can also extend the Cauchy integral formulas in superspace. If we define the invariant "distances," $\mathbf{z}_{ij} \equiv z_i - z_j - \theta_i \bar{\theta}_j - \bar{\theta}_i \theta_j, \theta_{ij} \equiv \theta_i - \theta_j, \bar{\theta}_{ij} \equiv \bar{\theta}_i - \bar{\theta}_j$, and the "volume" element $d\mathbf{z} = dz d\theta d\bar{\theta}$, then, [25],

$$
\begin{aligned}
&\frac{1}{2\pi i} \oint_c d\mathbf{z}_1\, \Phi(\mathbf{z}_1) \bar{\theta}_{12} \theta_{12} = 0 \\
&\frac{1}{2\pi i} \oint_c d\mathbf{z}_1\, \Phi(\mathbf{z}_1) \frac{\bar{\theta}_{12}\theta_{12}}{\mathbf{z}_{12}^m} = \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial z_2^{m-1}} \Phi(\mathbf{z}_2) \\
&\frac{1}{2\pi i} \oint_c d\mathbf{z}_1\, \Phi(\mathbf{z}_1) \frac{\bar{\theta}_{12}}{\mathbf{z}_{12}^m} = \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial z_2^{m-1}} D\Phi(\mathbf{z}_2) \\
&\frac{1}{2\pi i} \oint_c d\mathbf{z}_1\, \Phi(\mathbf{z}_1) \frac{\theta_{12}}{\mathbf{z}_{12}^m} = -\frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial z_2^{m-1}} \bar{D}\, \Phi(\mathbf{z}_2) \\
&\frac{1}{2\pi i} \oint_c d\mathbf{z}_1\, \Phi(\mathbf{z}_1) \frac{1}{\mathbf{z}_{12}^m} = \frac{1}{2} \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial z_2^{m-1}} [D, \bar{D}]\, \Phi(\mathbf{z}_2)
\end{aligned}
\tag{2.21}
$$

The prescription to evaluate the integrals above is the following: First, do the Grassman integrations using eq. (2.10) and then perform the complex integrations in the usual way. The contour C is winding around the point z_2.

The N=2 superanalytic transformations are generated by the energy-momentum superfield, which in component form can be written as:

$$
\mathbf{J}(\mathbf{z}) \equiv J(z) + i\theta \bar{G}(z) + i\bar{\theta} G(z) + 2\theta\bar{\theta} T(z).
\tag{2.22}
$$

The Fourier modes of the generators are defined in the usual way:

$$
\begin{aligned}
J(z) &\equiv \sum_{n \in \mathbf{Z}} \frac{J_n}{z^{n+1}} & T(z) &\equiv \sum_{n \in \mathbf{Z}} \frac{L_n}{z^{n+2}} \\
G(z) &\equiv \sum_{n \in \mathbf{Z}} \frac{G_{n-1/2}}{z^{n+1}} & \bar{G}(z) &\equiv \sum_{n \in \mathbf{Z}} \frac{\bar{G}_{n-1/2}}{z^{n+1}}
\end{aligned}
\tag{2.23}
$$

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These generators are represented in the space of superfield functions in the following way:

$$L_n \equiv -z^{n+1}\frac{\partial}{\partial z} - \frac{n+1}{2}z^n[\theta\frac{\partial}{\partial\theta} + \bar{\theta}\frac{\partial}{\partial\bar{\theta}}]$$

$$J_n \equiv z^n[\bar{\theta}\frac{\partial}{\partial\bar{\theta}} - \theta\frac{\partial}{\partial\theta}]$$

$$G_{n-1/2} \equiv z^n[\frac{\partial}{\partial\theta} - \bar{\theta}\frac{\partial}{\partial z}] + nz^{n-1}\theta\bar{\theta}\frac{\partial}{\partial\theta} \qquad .(2.24)$$

$$\bar{G}_{n-1/2} \equiv z^n[\frac{\partial}{\partial\bar{\theta}} - \theta\frac{\partial}{\partial z}] - nz^{n-1}\theta\bar{\theta}\frac{\partial}{\partial\bar{\theta}}$$

It is straightforward to check that the generators in eq. (2.24) satisfy the N=2 supercon-formal loop algebra, (as in (2.14) with $\tilde{c} = 0$), which is the algebra of N=2 superconformal transformations over S^1. The explicit representation (2.24) will be useful later on in this paper, to analyze the correlation functions of N=2 superconformal invariant theories.

The energy momentum tensor has an operator product expansion with itself:

$$\mathbf{J}(\mathbf{z}_1)\mathbf{J}(\mathbf{z}_2) = \frac{\theta_{12}}{\mathbf{z}_{12}}D\mathbf{J}(\mathbf{z}_2) - \frac{\bar{\theta}_{12}}{\mathbf{z}_{12}}\bar{D}\mathbf{J}(\mathbf{z}_2) + 2\frac{\theta_{12}\bar{\theta}_{12}}{\mathbf{z}_{12}^2}\mathbf{J}(\mathbf{z}_2)$$

$$+2\frac{\theta_{12}\bar{\theta}_{12}}{\mathbf{z}_{12}}\mathbf{J}'(\mathbf{z}_2) + \frac{\tilde{c}}{\mathbf{z}_{12}^2} \qquad (2.25)$$

where the anomaly $\tilde{c}$ is normalized, so that a free scalar N=2 superfields has $\tilde{c} = 1$. Eq. (2.25) corresponds to a change of the energy momentum tensor under a superconformal transformation

$$\delta_v\mathbf{J}(\mathbf{z}) = [\partial_z v]\mathbf{J}(\mathbf{z}) + v\partial_z\mathbf{J}(\mathbf{z}) + \frac{1}{2}[\bar{D}v]D\mathbf{J}(\mathbf{z}) + \frac{1}{2}[Dv]\bar{D}\mathbf{J}(\mathbf{z})$$

$$+\frac{\tilde{c}}{4}\partial_z[\bar{D}, D]v \qquad (2.26)$$

v being an infinitesimal N=2 superfield.

The change in the energy momentum tensor under a finite superconformal transformation is given by:

$$\mathbf{J}(\mathbf{z}) = \mathbf{J}'(\mathbf{z}')[D\theta'][\bar{D}\bar{\theta}'] + \frac{\tilde{c}}{2}S(\mathbf{z}, \mathbf{z}') \tag{2.27}$$

where the N=2 super-Schwarzian derivative is defined through:

$$S(\mathbf{z}, \mathbf{z}') \equiv \frac{\partial \bar{D}\bar{\theta}'}{\bar{D}\bar{\theta}'} - \frac{\partial D\theta'}{D\theta'} - 2\frac{\partial \bar{\theta}' \, \partial \theta'}{(\bar{D}\bar{\theta}') \, (D\theta')} \tag{2.28}$$

It satisfies the following composition law:

$$S(\mathbf{z}_1, \mathbf{z}_3) = S(\mathbf{z}_1, \mathbf{z}_2) + (D\theta_2)(\bar{D}\bar{\theta}_2)S(\mathbf{z}_2, \mathbf{z}_3) \tag{2.29}$$

On the sphere for a vector field to be globally defined, it must have a vanishing "anomaly," that is under an infinitesimal transformation generated by it, the anomalous part in (2.26) must vanish, that is, $\partial_z[\bar{D}, D]v = 0$, which gives an eight-parameter family of globally defined vector fields on the sphere:

$$v(\mathbf{z}) = v_{-1} + v_0 z + v_1 z^2 + \theta[u_{-1/2} + u_{1/2}z] + \bar{\theta}[\bar{u}_{-1/2} + \bar{u}_{1/2}z] + q_0\theta\bar{\theta}. \tag{2.30}$$

These vector fields generate the global N=2 superconformal algebra, $osp(2|2)$. (In fact they generate half of $osp(2|2)$, its holomorphic part.) The global N=2 superconformal algebra is the maximal, finite dimensional, subalgebra of the N=2 superconformal algebra. It contains the generators of the ordinary projective transformations, L_1, L_0, L_{-1}, the supercharges $G_{\pm 1/2}, \bar{G}_{\pm 1/2}$ and the zero mode of the U(1) current. It is easy to check using (2.14) that this set of generators closes into itself, and it contains as a subalgebra, the N=1 superconformal algebra, $osp(2|1)$. Since the Schwarzian derivative transforms as in (2.29), the fact that it vanishes for infinitesimal global N=2 transformations continues to be true for finite transformations belonging to the identity component of the group.

The $OSP(2|2)$ group transformations can be found either by exponentiating the generators of the algebra given in (2.24) or using the general form of superanalytic transformations (2.18), and some analyticity arguments [16]. Another way is to solve the equation $S(\mathbf{z}, \mathbf{z}') = 0$. There are three parameters associated with the subgroup $SL(2,C)$, four supersymmetry parameters, (Grassman), $\epsilon_1, \epsilon_2, \bar{\epsilon}_1, \bar{\epsilon}_2$ and a parameter q associated with the zero mode of the U(1) current. The group transformations are:

$$z' = \frac{az+b}{cz+d} + e^q\theta\frac{(1-\frac{1}{2}\epsilon_1\bar{\epsilon}_2)\bar{\epsilon}_1 z + \bar{\epsilon}_2(1+\frac{1}{2}\epsilon_2\bar{\epsilon}_1)}{(cz+d)^2} + e^{-q}\bar{\theta}\frac{(1+\frac{1}{2}\epsilon_1\bar{\epsilon}_2)\epsilon_1 z + (1-\frac{1}{2}\epsilon_1\bar{\epsilon}_2)\epsilon_2}{(cz+d)^2}$$

$$+\frac{[2d\epsilon_1\bar{\epsilon}_1 - 2c(\bar{\epsilon}_1\epsilon_2 + \bar{\epsilon}_2\epsilon_1)]z + d(\bar{\epsilon}_1\epsilon_2 + \bar{\epsilon}_2\epsilon_1) - 2c\bar{\epsilon}_2\epsilon_2}{(cz+d)^3} \qquad (2.31a)$$

$$\theta' = \frac{\epsilon_1 z + \epsilon_2}{cz+d} + e^q\theta\frac{1+\frac{1}{2}(\epsilon_2\bar{\epsilon}_1 - \epsilon_1\bar{\epsilon}_2) + \frac{1}{4}\epsilon_1\bar{\epsilon}_1\epsilon_2\bar{\epsilon}_2}{(cz+d)} + \theta\bar{\theta}\frac{\epsilon_1 d - \epsilon_2 c}{(cz+d)^2} \qquad (2.31b)$$

$$\bar{\theta}' = \frac{\bar{\epsilon}_1 z + \bar{\epsilon}}{cz+d} + e^{-q}\bar{\theta}\frac{1+\frac{1}{2}(\epsilon_2\bar{\epsilon}_1 - \epsilon_1\bar{\epsilon}_2) + \frac{1}{4}\epsilon_1\bar{\epsilon}_1\epsilon_2\bar{\epsilon}_2}{(cz+d)} + \theta\bar{\theta}\frac{\bar{\epsilon}_2 c - \bar{\epsilon}_1 d}{(cz+d)^2} \qquad (2.31c)$$

The N=2 superconformal vector field generates the group of N=2 super-diffeomorphisms on the circle, $\hat{\tilde{D}}iff(S^1)$. The Schwarzian derivative is the globally invariant generator of the second cohomology group of $\hat{\tilde{D}}iff(S^1)$. It generates a non-trivial transformation on the energy-momentum tensor viewed as a connection on moduli space.

As can be seen from (2.14), the subalgebra does not have an anomaly even if $\tilde{c} \neq 0$. This is of crucial importance in a superconformal theory as we will see later. It implies that all correlation functions are invariant under $OSP(2|2)$ constraining in such a way their form. Along with some supplementary constraints on the correlation functions, present when the theory has degenerate representations, it helps to determine the correlation functions completely, rendering the theory exactly solvable.

3. The General Structure of N=2 Superconformal Theories

An N=2 superconformal field theory is a field theory invariant under the N=2 superanalytic transformations described in the previous section, which form the N=2 superconformal group. The infinitesimal transformations are generated by an infinitesimal local superfield $v(\mathbf{z})$:

$$v(\mathbf{z}) \equiv v_0(z) + \theta\, v_1(z) + \bar{\theta}\bar{v}_1(z) + \theta\bar{\theta}\, v_2(z) \tag{3.1}$$

$$z' = z + v(\mathbf{z}) + \frac{1}{2}[(\bar{D}v)\theta + (Dv)\bar{\theta}]$$
$$\theta' = \theta + \frac{1}{2}\bar{D}v \quad , \quad \bar{\theta}' = \bar{\theta} + \frac{1}{2}Dv \tag{3.2}$$

The function $v_1, \bar{v}_1$ are Grassman functions anticommuting among themselves and with $\theta, \bar{\theta}$, whereas v_0, v_2 are usual meromorphic functions. The superconformal transformations are generated by the super-energy-momentum tensor, see (2.22). Using the Cauchy formulas of the previous section we can write the change of a local superfield under a superconformal transformation as:

$$\delta_v \Phi(\mathbf{z}) = -\frac{1}{4\pi i} \oint\limits_{C_z} d\mathbf{z}' v(\mathbf{z}')\mathbf{J}(\mathbf{z}')\Phi(\mathbf{z}) \tag{3.3}$$

where the contour C_z surrounds the point z in the complex plane.

The variation (3.3) is determined by the singularities of the OPE, of the energy-momentum tensor with the superfield. In particular a superfield function transforms under an infinitesimal transformation as:

$$\delta_v \Phi = v\partial\Phi + \frac{1}{2}Dv\bar{D}\Phi + \frac{1}{2}\bar{D}vD\Phi. \tag{3.4}$$

It is usually convenient to use radial quantizations going, (through a superanalytic transformation), from the cylinder to the plane, $(lnz, z^{-1/2}\theta, z^{-1/2}\bar{\theta}) \longleftrightarrow (\tau + i\sigma, \theta, \bar{\theta})$.

The fermionic fields on the cylinder can have two possible boundary conditions, periodic or antiperiodic. On the plane, this is translated to $G, \bar{G}(ze^{2\pi i}) = \pm G, \bar{G}(z)$, the corresponding subspaces of the full Hilbert spaces being the NS and R sectors. In the NS sector $G(ze^{2\pi i}) = G(z)$ whereas in the Ramond sector, $G(ze^{2\pi i}) = -G(z)$, that is the fermionic fields are double valued on the plane.

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The operator product expansion for the energy-momentum tensor was given in (2.25). The terms that appear in (2.25) are the most general terms that are allowed in a Euclidean N=2 supersymmetric quantum field theory, satisfying the standard constructive field theory axioms. The proof of [2] can be extended easily in our case, to guarantee (2.25) provided the theory has scale invariance and global N=2 supersymmetry. Using the mode expansions (2.23) we can derive (2.14) from (2.25). The energy momentum tensor must be a Hermitian operator, implying some hermiticity conditions among its components:

$$L_n^\dagger = L_{-n}, \ J_n^\dagger = J_{-n}, \ G_r^\dagger = \bar{G}_{-r}, \ \bar{G}_r^\dagger = G_{-r}. \tag{3.5}$$

We define the in-vacuum $|0\rangle$ of the theory at time $\tau = -\infty, (z = 0)$, to be $OSP(2|2)$ invariant. This means that it is annihilated by $L_n, n \geq -1, J_n, n \geq 0, G_r, \bar{G}_r, r \geq -1/2, (NS$ sector$)$ or $G_n, \bar{G}_n, n \geq 0$ in the R sector. In the same way the out-vacuum is defined at $z \to \infty$. The vacuum state belongs to the NS sector and it is the ground state of the theory. The unitary irreducible representations of the N=2 superconformal algebra are generated from highest weight vectors, (hwv), by the action of the lowering operators of the algebra , $L_n, J_n, G_r, \bar{G}_r, n, r < 0$.

In the NS sector the hwv's are generated by the action of primary conformal superfields on the vacuum state. Their defining relations are their transformation properties under superconformal transformations encoded in their OPE with the energy-momentum tensor:

$$\begin{aligned} \mathbf{J}(\mathbf{z_1})\Phi(\mathbf{z_2}) = {} & 2\Delta\frac{\theta_{12}\bar{\theta}_{12}}{\mathbf{z}_{12}^2}\Phi(\mathbf{z_2}) + 2\frac{\theta_{12}\bar{\theta}_{12}}{\mathbf{z}_{12}}\Phi'(\mathbf{z_2}) + \frac{\theta_{12}}{\mathbf{z}_{12}}D\Phi(\mathbf{z_2}) \\ & - \frac{\bar{\theta}_{12}}{\mathbf{z}_{12}}\bar{D}\Phi(\mathbf{z_2}) + \frac{Q\Phi(\mathbf{z_2})}{\mathbf{z}_{12}} \end{aligned} \tag{3.6}$$

Using (3.3) and (3.6) we can derive the transformation law for a primary superfield operator:

$$\begin{aligned} \delta_v\Phi(\mathbf{z}) = {} & \Delta(\partial_z v)\Phi(\mathbf{z}) + v\partial_z\Phi(\mathbf{z}) + \frac{1}{2}[\bar{D}v]D\Phi(\mathbf{z}) + \frac{1}{2}[Dv]\bar{D}\Phi(\mathbf{z}) \\ & - \frac{Q}{4}\{[D,\bar{D}]v\}\Phi(\mathbf{z}) \end{aligned} \tag{3.7}$$

Under a finite transformation $\Phi(\mathbf{z})$ transforms as:

$$\Phi(\mathbf{z}) = \Phi(\mathbf{z}')[D\theta']^{\Delta+\frac{Q}{2}}[\bar{D}\bar{\theta}']^{\Delta-\frac{Q}{2}} \tag{3.8}$$

where (Δ, Q) are its dimension and $U(1)$ charge. The *hwv* in the *NS* sector are characterized by their eigenvalues under the zero modes of the algebra:

$$L_0|\Phi\rangle = \Delta|\Phi\rangle \qquad , \qquad J_0|\Phi\rangle = Q|\Phi\rangle. \tag{3.9}$$

Being *hw* states they must be annihilated by the raising operators of the algebra:

$$L_n|\Phi\rangle = J_n|\Phi\rangle = G_n|\Phi\rangle = \bar{G}_n|\Phi\rangle, n > 0. \tag{3.10}$$

The OPE (3.6) can be written also as commutation relations which will be useful later on:

$$
\begin{aligned}
[L_n, \Phi(\mathbf{z})] &= z^{n+1}\frac{\partial}{\partial z}\Phi(\mathbf{z}) + (n+1)z^n[\Delta + \frac{1}{2}[\theta\frac{\partial}{\partial\theta} + \bar{\theta}\frac{\partial}{\partial\bar{\theta}}]]\Phi(\mathbf{z}) \\
&\quad + \frac{Q}{2}n(n+1)z^{n-1}\theta\bar{\theta}\Phi(\mathbf{z}) \\
[J_n, \Phi(\mathbf{z})] &= z^n[Q + \bar{\theta}\frac{\partial}{\partial\bar{\theta}} - \theta\frac{\partial}{\partial\theta}]\Phi(\mathbf{z}) + 2n\Delta z^{n-1}\theta\bar{\theta}\Phi(\mathbf{z}) \\
[G_r, \Phi(\mathbf{z})] &= z^{r+\frac{1}{2}}[\frac{\partial}{\partial\bar{\theta}} - \theta\frac{\partial}{\partial z}]\Phi(\mathbf{z}) - (r+\frac{1}{2})z^{r-\frac{1}{2}}[(2\Delta+Q)\theta + \theta\bar{\theta}\frac{\partial}{\partial\bar{\theta}}]\Phi(\mathbf{z}) \\
[\bar{G}_r, \Phi(\mathbf{z})] &= z^{r+\frac{1}{2}}[\frac{\partial}{\partial\theta} - \bar{\theta}\frac{\partial}{\partial z}]\Phi(\mathbf{z}) - (r+\frac{1}{2})z^{r-\frac{1}{2}}[(2\Delta-Q)\bar{\theta} - \theta\bar{\theta}\frac{\partial}{\partial\bar{\theta}}]\Phi(\mathbf{z})
\end{aligned}
\tag{3.11}
$$

In the R-sector the zero modes are L_0, J_0, and $\bar{G}_0, G_0$, their eigenvalues characterizing *hwv*'s. There are two kinds of *hwv*'s, $|\Delta, Q \mp 1/2\rangle_\pm$, [9],

$$
\begin{aligned}
L_0|\Delta, Q \mp \frac{1}{2}\rangle_\pm &= \Delta|\Delta, Q \mp \frac{1}{2}\rangle_\pm \\
J_0|\Delta, Q \mp \frac{1}{2}\rangle_\pm &= (Q \mp \frac{1}{2})|\Delta, Q \mp \frac{1}{2}\rangle_\pm
\end{aligned}
\tag{3.12}
$$

which satisfy an additional *hwv* condition with respect to the supercharges:

$$G_0|\Delta, Q + \frac{1}{2}\rangle_- = 0 \quad , \quad \bar{G}_0|\Delta, Q - \frac{1}{2}\rangle_+ = 0. \tag{3.13}$$

Consequently there are two kinds of representations, $R^\pm$. The two representations are isomorphic under charge conjugations $(G_n \leftrightarrow \bar{G}_n, J_n \to -J_n)$.

¿From now on we will restrict to one of them, say R^+, our statements being valid for R^- as well.

In the R-sector the ground state is not unique. There are two ground states degenerate in energy, (i.e., having the same dimension). $|\Theta^+\rangle$ and $G_0|\Theta^+\rangle \equiv |\Theta^-\rangle$. They are generated from the vacuum $|0\rangle$, (which belongs to the NS sector), by primary fields $\Theta^\pm(\mathbf{z})$, much like the spin fields of the N=1 superconformal theories. The spin fields have double-valued OPE with the energy-momentum tensor, for example:

$$G(z)\Theta^\pm(\omega) = \frac{1}{2}\alpha_\pm \frac{\Theta^\mp(\omega)}{(z-\omega)^{3/2}} \qquad (3.14)$$

where $\alpha_+ = 1, \quad \alpha_- = \Delta - \tilde{c}/8$. This happens in order for the spin field to be able to change the boundary conditions of the fermionic parts of the superfields. We can view the spin fields as opening and closing cuts on the cylinder. The states in the R-sector are generated by ordinary conformal superfields acting on the Ramond ground states. The generators of global N=2 supersymmetry transformations in the R-sector are $G_0, \bar{G}_0$.

Unbroken N=2 supersymmetry is implied by the existence of a ground state which is annihilated by the global N=2 supersymmetry generators. The state $|\Theta^+\rangle$ is annihilated by $\bar{G}_0$ due to (3.13). Applying $\{G_0, \bar{G}_0\}$ to it we obtain:

$$\{G_0, \bar{G}_0\}|\Theta^+\rangle = \bar{G}_0 G_0|\Theta^+\rangle = (2L_0 - \tilde{c}/4)|\Theta^+\rangle = (2\Delta - \tilde{c}/4)|\Theta^+\rangle. \qquad (3.15)$$

Consequently, in order for G_0 to annihilate $|\Theta^+\rangle$, its dimension must be $\Delta_+ = \tilde{c}/8$. The operator $\{G_0, \bar{G}_0\}$ is a hermitian positive operator, thus any dimension in the R-sector has to be $\geq \tilde{c}/8$. This is the reason that the vacuum $|0\rangle$, the lowest energy state must belong to the NS-sector. In the same way $\bar{G}_0|\Theta^-\rangle = 0$, implies $\Delta_- = \tilde{c}/8$. Therefore, the existence of a state in the R-sector with $\Delta = \tilde{c}/8$ implies unbroken N=2 supersymmetry on the cylinder. On the other hand if such a state does not exist in the theory the one supersymmetry out of the two is broken.

So far we have been discussing the two sectors of the N=2 superconformal theory that parallel the situation in ordinary N=1 superconformal theories. In the N=2 case though, unlike the

N=1, there is another sector present in general due to the fact that N=2 superfields contain two fermionic components, so there is also the possibility of choosing periodic boundary conditions for one of them, and antiperiodic for the other one. This can be seen easier if we write the algebra (2.14) in an O(2) basis:

$$G_r^1 \equiv \frac{G_r + \bar{G}_r}{\sqrt{2}} \quad , \quad G_r^2 = \frac{G_r - \bar{G}_r}{i\sqrt{2}}. \tag{3.16}$$

In this basis the algebra (2.14) becomes:

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{\tilde{c}}{4}(m^3 - m)\delta_{m+n,0} \\
[L_m, G_r^i] &= (\frac{m}{2} - r)G_{m+r}^i, \quad [L_m, J_n] = -nJ_{m+n} \\
[J_m, J_n] &= \tilde{c}m\delta_{m+n,0} \quad [J_m, G_r^i] = i\epsilon^{ij}G_{m+r}^j \\
\{G_r^i, G_s^j\} &= 2\delta^{ij}L_{r+s} + i\epsilon^{ij}(r-s)J_{r+s} + \tilde{c}(r^2 - \frac{1}{4})\delta^{ij}\delta_{r+s,0}
\end{aligned}
\tag{3.17}
$$

The twisted (T) N=2 algebra is defined by choosing integer modes for G_m^1, L_m and half integer modes for G_r^2, J_r choices, compatible with the commutation relations (3.17). In the O(2) basis the energy momentum tensor becomes:

$$\mathbf{J}(\mathbf{z}) \equiv J(z) + \epsilon^{ij}\theta^i G^j(z) + \epsilon^{ij}\theta^i\theta^j T(z) \tag{3.18}$$

where θ^i is an O(2) doublet of Grassmann coordinates. A twisted superfield:

$$\Phi(\mathbf{z}) \equiv \phi(z) + \epsilon^{ij}\theta^i\psi^j(z)(z) + \frac{1}{2}\epsilon^{ij}\theta^i\theta^j g(z) \tag{3.19}$$

has antiperiodic boundary conditions for $\phi(z)$ and $\psi^2(z)$ and periodic boundary conditions for $g(z)$ and $\psi^1(z)$, on the cylinder, that is ϕ and ψ^1 are $\mathbf{Z}_2$ twisted. Again here, G_0^1 is a hermitian operator. Its square, acting on a primary state must give positive eigenvalues, which implies that all the dimensions in the T-sector satisfy: $\Delta \geq \tilde{c}/8$. In particular it implies that if there is a state with $\Delta = \tilde{c}/8$ this is then the ground state, and it is doubly degenerate since this state

$|H^+\rangle$ and $|H^-\rangle \equiv G_0^1|H^+\rangle$, have the same energies. One of the two supersymmetries, namely the one generated by G_0^1 is then unbroken, since G_0^1 annihilates the ground states:

$$(G_0^1)^2|H^+\rangle = \frac{1}{2}\{G_0^1, G_0^1\}|H^+\rangle = (L_0 - \tilde{c}/8)|H^+\rangle = 0$$
$$G_0^1|H^-\rangle = (G_0^1)^2|H^+\rangle = 0 \tag{3.20}$$

The global supersymmetry generated by $G_{-1/2}^2$ is broken since $G_{-1/2}^2$ fails to annihilate the ground states. This is obvious since in order for $G_{1/2}^2$ to annihilate a primary state, its dimension has to be zero, and as we argued above, states with zero dimension do not exist in the T-sector. Thus in the T-sector we have at most a remnant N=1 supersymmetry. The ground states are generated from the NS vacuum by the "twist" fields $H^\pm(\mathbf{z})$, the presence of which induces cuts on the complex plane such that $\phi(z)$ and $\psi^1(z)$ are double valued around the point where the twist field lies. In the T-sector there is a parity operator, $(-1)^F$, which commutes with L_m, J_m and anticommutes with G_m^i. In particular:

$$(-1)^F|H^+\rangle = |H^+\rangle, \; (-1)^F|H^-\rangle = -|H^-\rangle, \tag{3.21}$$

In the R-sector the two-spin fields are non-local with respect to each other. Their operator product expansion contains square root singularities in the complex plane which induce non-locality when we project to Euclidean space. The same is true in the T-sector. In order to obtain a local theory we must suitably project out one fermion parity, the same way as in the N=1 case.

Unitarity, as in N=0,1 conformal theories, puts severe constraints in the representation content of an N=2 superconformal theory. A basic tool for studying unitarity is the Kač determinant. It has been derived in [9, 10, 11]. We will include the main results, since they will be useful later on, in this paper.

In the NS sector hwv's are labeled by their dimension h and charge q. Any secondary state is then characterized by its level, (eigenvalue of $L_0 - h$), and relative charge, (eigenvalue of

$J_0 - q$). The Kač determinant at level n and relative charge m is given by:

$$det M_{n,m}^{NS}(\tilde{c}, h, q) = \prod_{\substack{1 \leq rs \leq 2n \\ s, even}} [f_{r,s}^{NS}]^{P_{NS}(n-\frac{rs}{2}, m)} \prod_{k \in \mathbf{Z}+1/2} [g_k^{NS}]^{\tilde{P}_{NS}(n-|k|, m-sgn(k); k)} \tag{3.22}$$

$$F_{r,s}^{NS} = 2(\tilde{c}-1)h - q^2 - \frac{(\tilde{c}-1)^2}{4} + \frac{[(\tilde{c}-1)^2 + s]^2}{4}, r \in \mathbf{Z}^+, s \in 2\mathbf{Z}^+ \tag{3.23a}$$

$$g_k^{NS} = 2h - 2kq + (\tilde{c}-1)(k^2 - \frac{1}{4}), k \in \mathbf{Z} + \frac{1}{2} \tag{3.23b}$$

while the NS partitions function are defined by:

$$\sum_{n,m} P_{NS}(n, m) z^n w^m = \prod_{k=1}^{\infty} \frac{(1 + z^{k-\frac{1}{2}}w)(1 + z^{k-\frac{1}{2}}w^{-1})}{(1 - z^k)^2} \tag{3.24a}$$

$$\sum_{n,m} \tilde{P}_{NS}(n, m; k) z^n w^m = [1 + z^{|k|}w^{sgn(k)}]^{-1} \sum_{n,m} P_{NS}(n, m) z^n w^m \tag{3.24b}$$

where z, w are formal complex variables. Equation (3.22) implies that whenever there is a vanishing of $f_{r,s}^{NS}$, there exists a hwv at level $\frac{rs}{2}$ and relative charge zero. When $g_k^{NS} = 0$, there is a hwv at level $|k|$ and relative charges $sgn(k)$. For $\tilde{c} < 1$ unitary representations exist only for the discrete series of values for $\tilde{c}$;

$$\tilde{c} = 1 - \frac{2}{m}, \qquad m = 2, 3, u \ldots \tag{3.25}$$

their dimension and charges are given by:

$$h = \frac{4jk - 1}{4m}, \qquad q = \frac{j-k}{m}, \qquad j, k \epsilon \mathbf{Z} + \frac{1}{2}, \qquad 0 < j, k, j+k \leq m-1 \tag{3.26}$$

Representations belonging to this class, (NS_0), are degenerate. For $\tilde{c} \geq 1$ there are two other classes of unitary representations.

(a) NS_2. These representations are characterized by the following conditions:

$$g_n^{NS} = 0, \quad g_{n+sgn(n)}^{NS} < 0, \quad f_{1,2}^{NS} > \rho \ \text{for some } n \in \mathbf{Z} + \frac{1}{2} \tag{3.27}$$

They are also degenerate.

(b) NS_3. The representation in this class are characterized by $g_n^{NS} \geq 0, \forall n\epsilon\mathbf{Z} + \frac{1}{2}$. They contain degenerate representations when $g_n^{NS} = 0$ for some $n\epsilon\mathbf{Z} + \frac{1}{2}$.

In the $R^\pm$ sector $hwv's$ are characterized by their dimension h and charge $q \pm 1/2$. They satisfy also the supplementary hwv conditions (3.13). The Kač determinate at level n and relative charge m is:

$$\det M_{n,m}^R(\tilde{c}, h, q) = \prod_{\substack{1 \leq rs \leq 2n \\ s, even}} [f_{r,s}^R]^{P_R(n-\frac{rs}{2},m)} \prod_{k\epsilon\mathbf{Z}} [g_k^R]^{\tilde{P}_R(n-|k|,m-sgn(k);k)} \tag{3.28}$$

$$f_{r,s}^R = 2(\tilde{c} - 1)(h - \tilde{c}/8) - q^2 + \frac{[(\tilde{c}-1)r + s]^2}{4}, r\epsilon\mathbf{Z}^+, s\epsilon 2\mathbf{Z}^+ \tag{3.29a}$$

$$g_k^R = 2h - 2kq + (\tilde{c} - 1)(k^2 - \frac{1}{4}) - \frac{1}{4}, k\epsilon\mathbf{Z} \tag{3.29b}$$

where the Ramond partition functions are defined by:

$$\sum_{n,m} P_R(n,m)z^n w^m = (w^{1/2} + w^{-1/2}) \prod_{k=1}^{\infty} \frac{(1 + z^k w)(1 + z^k w^{-1})}{(1 - z^k)^2} \tag{3.30a}$$

$$\sum_{n,m} \tilde{P}_R(n,m;k)z^n w^m = [1 + z^{|k|}w^{sgn(k)}]^{-1} \sum_{n,m} P_R(n,m)z^n w^m \tag{3.30b}$$

$sgn(k) = 1, \quad k > 0, \quad -1 \text{ for } k < 0, \quad \text{and} \quad sgn(0) = 1(-1) \text{ for the } R^+, (R^-) \text{ algebra}$ respectively.

A vanishing of $f_{r,s}^R$ signals the existence of a *hwv* at level $rs/2$ and relative charge $-1/2$. When $g_k^R = 0$ there is a *hwv* at level $|k|$ and relative charge $sgn(k) - \frac{1}{2}sgn(0)$. For $\tilde{c} < 1$ unitary representations exist only for the values given in (3.25), the respective dimensions and (1) charges being

$$h = \frac{\tilde{c}}{8} + \frac{jk}{m}, q = sgn(0)\frac{j-k}{m}, j, k\epsilon\mathbf{Z}, 0 = \leq j - 1, k, j + k \leq m - 1 \qquad (3.31)$$

This class of representations, $(R_0^{\pm})$, contains only degenerate representations. For $\tilde{c} \leq 1$, there are again two classes of unitary representations.

(a) $R_2^{\pm}$. The representations in this class are characterized by,

$$g_n^R = 0, g_{n+sgn(n)}^R < 0, f_{1,2}^R \geq 0, \qquad (3.32)$$

for some $n\epsilon\mathbf{Z}$. They are also degenerate.

(b) $R_3^{\pm}$. These representations are characterized by $g_n^R \geq 0, \forall n\epsilon\mathbf{Z}$.

They contain a subset of degenerate representations corresponding to $g_n^R = 0$ for some $n\epsilon\mathbf{Z}$.

Finally in the T-sector the *hwv* are characterized by their dimension h and fermion parity, $(-1)^F$. Each level contains two equal subspaces of opposite fermion parity. The Kač determinant for the T-algebra is the following:

$$\det M_{+0}^T = 1, \det M_{-0}^T = h - \tilde{c}/8 \qquad (3.33a)$$

$$\det M_{\pm n}^T(\tilde{c}, h) = [h - \tilde{c}/8]^{P_T(n)/2} \prod_{\substack{1 \leq rs \leq 2n \\ s, odd}} [f_{r,s}^T]^{P_T(n-rs/2)} \qquad (3.33b)$$

$$f_{r,s}^T = 2(\tilde{c} - 1)(h - \tilde{c}/8) + \frac{[(\tilde{c} - 1)r + s]^2}{4}, s\epsilon 2\mathbf{Z}^+ - 1 \qquad (3.34a)$$

$$\sum_n P_T(n)z^n = \prod_{k=1}^{\infty} \frac{(1 + z^k)(1 + z^{k-1/2})}{(1 - z^k)(1 - z^{k-1/2})} \qquad (3.35)$$

For $\tilde{c} < 1$ unitary representations exist again for the values (3.25), all being degenerate, with

dimensions given by:

$$h = \frac{\tilde{c}}{8} + \frac{(m-2r)^2}{16m}, m = 2, 3, \ldots, r\epsilon\mathbf{Z}, 1 \leq r \leq m/2 \tag{3.36}$$

For $\tilde{c} \geq 1$, we have another class of unitary representations, T_2, with $h \geq \tilde{c}/8$. None of them is degenerate.

The degenerate representations belonging to the class $NS_0, R_0^{\pm}, T_0$, have been proven to be unitary through an explicit unitary construction of their Hilbert space, [17]. The question about the unitary of $R_2^{\pm}$, and NS_2 is still open.

4. Correlation Functions and Operator Algebra of the Unitary Degenerate Representations

As it was mentioned earlier in this work, the invariance of the vacuum under the global $N = 2$ superconformal group, $OSP(2|2)$, turns out to be very useful towards the evaluation of the correlation functions[*] ¿From now on we restrict ourselves to the NS sector. Similar techniques though apply to the $R^{\pm}$ and T sectors although the analysis is somewhat more complicated. As an example we present the evaluation of the two-point functions in the R and T sectors in appendix A.

Using the commutations relations (3.11), derived in the previous section, we can write the Ward identities for global superconformal invariance. Their derivation is obvious. For example L_{-1} annihilates the in-vacuum. But we can move it to the left using (3.11), so we end up with a differential equation for the correlation function. Thus the n-point function:

$$F_n \equiv \langle 0|\Phi_1(\mathbf{z}_1)\Phi_2(\mathbf{z}_2)\Phi_3(\mathbf{z}_3)\ldots\Phi_n(\mathbf{z}_n)|0\rangle$$

---

[*] The invariance under $OSP(2|2)$ is still present even when the system lies on a curved manifold where the anomaly $\tilde{c}$ couples to the Ricci scalar, $T_\alpha^\alpha \sim \tilde{c}R$, and local scale invariance as well as local supersymmetry are not symmetries any more.

satisfies the following Ward identities:

$$L_{-1} : \left[\sum_{i=1}^{n} \frac{\partial}{\partial z_i}\right] F_n = 0$$

$$L_0 : \sum_{i=1}^{n} \left\{ z_i \frac{\partial}{\partial z_i} + \Delta_i + \frac{1}{2}[\theta_i \frac{\partial}{\partial \theta_i} + \bar{\theta}_i \frac{\partial}{\partial \bar{\theta}_i}] \right\} F_n = 0 \qquad (4.1)$$

$$L_{+1} : \sum_{i=1}^{n} \left\{ z_i^2 \frac{\partial}{\partial z_i} + 2z_i \left\{ \Delta_i + z_i(\theta_i \frac{\partial}{\partial \theta_i} + \bar{\theta}_i \frac{\partial}{\partial \bar{\theta}_i}) \right\} + Q_i \theta_i \bar{\theta}_i \right\} F_n = 0$$

$$J_0 : \sum_{i=1}^{n} \left(Q_i + \bar{\theta}_i \frac{\partial}{\partial \bar{\theta}_i} - \theta_i \frac{\partial}{\partial \theta_i} \right) F_n = 0$$

$$G_{-1/2}, \bar{G}_{-1/2} : \sum_{i=1}^{n} \left(\frac{\partial}{\partial \bar{\theta}_i} - \bar{\theta}_i \frac{\partial}{\partial z_i} \right) F_n = \sum_{i=1}^{n} \left(\frac{\partial}{\partial \theta_i} - \bar{\theta}_i \frac{\partial}{\partial z_i} \right) F_n = 0$$

$$G_{1/2} : \sum_{i=1}^{n} \left[z_i[\frac{\partial}{\partial \bar{\theta}_i} - \theta_i \frac{\partial}{\partial z_i}] - (2\Delta_i + Q_i)\theta_i - \theta_i \bar{\theta}_i \frac{\partial}{\partial \theta_i} \right] F_n = 0 \qquad (4.2)$$

$$\bar{G}_{1/2} : \sum_{i=1}^{n} \left[z_i[\frac{\partial}{\partial \theta_i} - \bar{\theta}_i \frac{\partial}{\partial z_i}] - (2\Delta_i - Q_i)\bar{\theta}_i + \theta_i \bar{\theta}_i \frac{\partial}{\partial \theta_i} \right] F_n = 0$$

where Δ_i, Q_i are dimensions and charges of the various fields appearing in the correlation function (4.1).

A superfield operator in terms of components has the form:

$$\Phi(\mathbf{z}) \equiv \phi(z) + \theta \psi(z) + \bar{\theta} \psi(z) + \theta \bar{\theta} g(z). \qquad (4.3)$$

The two-point function is completely fixed by the Ward identities, up to an irrelevant normalization constant.

$$\langle 0|\Phi_1(\mathbf{z}_1)\Phi_2(\mathbf{z}_2)|0\rangle = \mathbf{z}_{12}^{-(\Delta_1+\Delta_2)} \exp\{Q_2 \frac{\theta_{12}\bar{\theta}_{12}}{\mathbf{z}_{12}}\}\delta_{Q_1+Q_2,0}\delta_{\Delta_1,\Delta_2} \qquad (4.4)$$

It is a function of the supersymmetry invariant distances in super space, $\mathbf{z}_{12} = z_1 - z_2 - \theta_1\bar{\theta}_2 - \bar{\theta}_1\theta_2$, $\theta_{12} = \theta_1 - \theta_2$, $\bar{\theta}_{12} = \bar{\theta}_1 - \bar{\theta}_2$. The three-point function depends on nine

independent variables $(z_i, \theta_i, \bar{\theta}_i)$. Since $OSP(2|2)$ has eight generators we can fix at most eight of them, so there must be a unique combination invariant under $OSP(2|2)$. This is a commuting combination which turns out to be nilpotent:

$$\hat{R} = \frac{\theta_{12}\bar{\theta}_{12}}{\mathbf{z}_{12}} - \frac{\theta_{13}\bar{\theta}_{13}}{\mathbf{z}_{13}} + \frac{\theta_{23}\bar{\theta}_{23}}{\mathbf{z}_{23}}, \hat{R}^2 = 0 \tag{4.5}$$

So, for any particular solution of the Ward identities, we can obtain the general solution by multiplying it with $(1 + \alpha\hat{R})$, α being an arbitrary commuting constant. Solving the Ward identities for the three-point function we obtain:

$$\langle 0|\Phi_1(\mathbf{z}_1)\Phi_2(\mathbf{z}_2)\Phi_3(\mathbf{z}_3)|0\rangle = C \left[\prod_{i<j}^{3} \mathbf{z}_{ij}^{-\Delta_{ij}}\right] \exp\left[\sum_{i<j}^{3} A_{ij}\frac{\theta_{ij}\bar{\theta}_{ij}}{\mathbf{z}_{ij}}\right] \delta_{Q_1+Q_2+Q_3,0} \tag{4.6}$$

where the constants

$$A_{ij} = -A_{ji}, \qquad \sum_{\substack{j=1 \\ j\neq i}}^{3} A_{ij} = -Q_i \tag{4.7}$$

It is easily seen from (4.7), that the equations defining the constants A_{ij}, are not fixing all of them because of the change neutrality condition, for the correlation function. In particular, if A_{ij} is some solution of (4.7) then $A_{12} + \alpha$, $A_{31} + \alpha$, $A_{23} + \alpha$, is also a solution. Of course this is expected. It corresponds to multiplying the three-point function by the $OSP(2|2)$ invariant, $(1 + \alpha\hat{R})$. For the three-point function to be non-zero, the OPE of the operators Φ_1, Φ_2 must contain the family Φ_3. Then the normalization constant C of the three-point function is the Glebsch-Gordan coefficient for the decomposition $[\Phi_1] \otimes [\Phi_2] \rightarrow [\Phi_3]$. In the $N = 2$ case, like the $N = 1$, there is another operator product coefficient to be determined, namely one of the A_{ij}, due to the existence of the $OSP(2|2)$ invariant $\hat{R}$.

In general $OSP(2|2)$ invariance constraints the n-point function to have the form:

$$\langle 0|\Phi_1(\mathbf{z}_1)\Phi_2(\mathbf{z}_2), \dots \Phi_n(\mathbf{z}_n)|0\rangle \sim \prod_{i<j}^{n} [\mathbf{z}_{ij}^{-\Delta_{ij}}] \exp[\sum_{i<j}^{n} A_{ij}\frac{\theta_{ij}\bar{\theta}_{ij}}{\mathbf{z}_{ij}}]$$

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$$\times F_n[x_1, x_2, \ldots, x_{3n-8}] \times \delta_{\sum_{i=1}^n Q_i, 0} \qquad (4.8)$$

$$A_{ij} = -A_{ji}, \qquad \Delta_{ij} = \Delta_{ji}, \qquad \sum_{\substack{j=1 \\ i \neq j}}^n A_{ij} = -Q_i, \qquad \sum_{\substack{i=1 \\ j \neq i}}^n \Delta_{ij} = 2\Delta_i \qquad (4.9)$$

where $x_i, i = 1, 2 \ldots, 3n - 8$ are the combinations of the coordinates, with dimension zero, invariant under $OSP(2|2)$. They are functions of the invariant distances, $\mathbf{z}_{ij}, \theta_{ij}, \bar{\theta}_{ij}$. All the non-trivial information about the theory is encoded in the functions F_n. In most cases they are determined by the specific details of the theory. In certain cases though, that we will discuss below, they can be evaluated, just by knowing the representation content of the theory.

So let's focus on these interesting situations where there are more constraints on the form of the correlation functions.

Consider a hw unitary irreducible representation of the $N = 2$ superconformal algebra. It is generated by a $hwv, |\Delta, Q >$, the primary state, satisfying the usual hwv conditions. The full representation is obtained from $|\Delta, Q >$ by applying the lowering operations of the algebra. In some special situations it may turn out that one of the secondary states satisfies the hwv conditions. That means that the representation generated by $|\Delta, Q >$ is not irreducible, but there is another representation, (the one generated by the secondary vector), embedded in it. The secondary $hwv, |\chi >$, has the interesting property, that it is null, (i.e. $(\chi|\chi) = 0$), and orthogonal to any other state in the Hilber space. We may thus consistently set $|\chi >$ to be equal to zero, a condition that decouples all its family from the correlation function of the theory. In fact this condition will generate constraints on the correlation functions, of the primary state $|\Delta, Q >$. To see how such constraints arise we have to remember that $|\chi >$ is given by some operator $\hat{O}$, constructed out of the lowering operators of the algebra, acting on $|\Delta, Q >$, thus:

$$0 \equiv < 0|\Phi_1(\mathbf{z}_1)\Phi_2(\mathbf{z}_2)\ldots\Phi_{n-1}(\mathbf{z}_{n-1})|\chi > = < 0|\Phi_1(\mathbf{z}_1)\Phi_2(\mathbf{z}_2)\ldots\Phi_{n-1}(\mathbf{z}_{n-1})\hat{O}|\Delta, Q > \quad (4.10)$$

Moving the operator $\hat{O}$ to the left using the commutation relations (3.11) we end up with a super-differential equation for the correlation function. Solving these equations we can determine

all the correlation functions that the degenerate family is participating in. An interesting set of theories are those that contain only degenerate representations, the so called "minimal" theories. They contain a discrete , (in general infinite) set of primary operators and exist for $\tilde{c} < 1$[*] A subset of them are unitary and their content has presented in section 3 according to the results of [9,10,11]. They contain a finite set of primary fields. Such theories are exactly solvable in the previous sense. Unitary minimal theories are known for systems realizing conformal, [1, 18], or superconformal invariance [19, 20]. The "fusion" rules in the N=0 case were derived in [1]. In the N=1 case they were partially derived in [19] and in full generality in [27].

A necessary and sufficient condition for the existence of such systems is the closure of the operator algebra of a set of unitary degenerate representations. In fact we will show that the operator algebra of the unitary degenerate representations of the $N = 2$ superconformal algebra, with $\tilde{c} < 1$, does close. We will derive also the "fusion" rules for the operator algebra.

Consider the OPE of two primary operators:

$$\Phi_1(\mathbf{z})\Phi_2(0) = \sum_i \Phi_i(0)\mathbf{z}^{\Delta_1 + \Delta_2 - \Delta_3} \qquad (4.11)$$

where the notation in the right hand side of (4.11) is symbolic, meaning the product can be written as a sum of primary operators and/or their descendants, and the $(z, \theta, \bar{\theta})$ dependence can be easily substituted back. What we want to know is which irreducible representation can appear in the operator product of two given representations. There is a simple criterion for representations which are not allowed, and this is the vanishing of the appropriate 3-point function.

The strategy is to use the superdifferential equations stemming from the degeneracy of the representations to derive selection rules for the operator product algebra. Let's consider a concrete example. Take a representation which has a null vector at the first level. Such a

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[*] It is possible that minimal theories exist also for $\tilde{c} > 1$ in the N=2 case, but we will not examine this question further in this paper.

representation is for example one with $\Delta = \frac{m-2}{2m}, Q = -\frac{m-2}{m}$, when $\tilde{c} = 1 - \frac{2}{m}, m = 2, 3 \dots$. The null vector at level one is given by:

$$|\chi_1^0 >= [(Q-1)L_{-1} - (2\Delta + 1)J_{-1} + G_{-1/2}\bar{G}_{-1/2}]|\Delta, Q > \qquad (4.12)$$

It is easy to verify, using the commutation relations (2.14), that $|\chi_1^0 >$ satisfies all the *hwv* conditions. Consider now the n-point function where this state is participating in. We've mentioned already that such a correlation function is identically zero.

$$0 \equiv \langle 0|\Phi_1(\mathbf{z}_1)\Phi_2(\mathbf{z}_2)\dots\Phi_n(\mathbf{z}_n)|\chi_1^0\rangle = \langle 0|\Phi_1(\mathbf{z}_1)\Phi_2(\mathbf{z}_2)\dots\hat{O}\Phi(0)|0\rangle \qquad (4.13)$$

Commuting $\hat{O}$ through to the left we arrive at the following superdifferential equation

$$\left\{(1-Q)\sum_{n=1}^{n}\frac{\partial}{\partial z_i} + (2\Delta + 1)\sum_{i=1}^{n}\left[\frac{Q_i}{z_i} + \frac{1}{z_i}(\bar{\theta}_i\frac{\partial}{\partial\theta_i} - \theta_i\frac{\partial}{\partial\theta_i}) - \frac{2\Delta_i}{z_i^2}\theta_i\bar{\theta}_i\right]\right.$$

$$\left. + \sum_{i=1}^{n}\sum_{j=1}^{n}\left[\frac{\partial}{\partial\bar{\theta}_i} - \theta_i\frac{\partial}{\partial z_i}\right]\left[\frac{\partial}{\partial\theta_j} - \bar{\theta}_i\frac{\partial}{\partial z_i}\right]\right\} < 0|\Phi(\mathbf{z}_1)\dots\Phi_n(\mathbf{z}_n)\Phi(0)|0 >= 0 \qquad (4.14)$$

We will specialize (4.14) to the 3-point function $< 0|\Phi_1(\mathbf{z}_1)\Phi_2(\mathbf{z}_2)\Phi_3(\mathbf{z}_3)|0 >$ where Φ_3 is the degenerate operator mentioned above.

Doing a translation and two global supersymmetry transformations (we have the freedom to do that, thanks to the $OSP(2|2)$ invariance of the correlation function), we can write the three-point function in the form $< 0|\Phi_1(\tilde{\mathbf{z}}_1)\Phi_2(\mathbf{z}_2)\Phi_3(0)|0 >$, where:

$$\tilde{\mathbf{z}}_1 \equiv (z_1 - z_3 - \theta_1\bar{\theta}_3 - \bar{\theta}_1\theta_3, \theta_1 - \theta_3, \bar{\theta}_1 - \bar{\theta}_3)$$

$$\tilde{\mathbf{z}}_2 \equiv (z_2 - z_3 - \theta_2\bar{\theta}_3 - \bar{\theta}_2\theta_3, \theta_2 - \theta_3, \bar{\theta}_2 - \bar{\theta}_3) \qquad (4.15)$$

Using the form of the three-point function found earlier, in (4.14) we arrive at the following

set of conditions for the dimension Δ_{ij} and the constants A_{ij}:

$$\Delta_{13}(Q_3 - 1) + Q_1(2\Delta_3 + 1) + A_{13} + \Delta_{13} = 0$$

$$(\Delta_{12} - A_{12})(A_{13} + \Delta_{13}) = 0, \qquad (\Delta_{12} + A_{12})(A_{13} + \Delta_{13}) = 0$$

$$(Q_3 - 1)A_{13} - 2\Delta_1(2\Delta_3 + 1) + (\Delta_{13} - A_{13} + 1)(\Delta_{13} + A_{13}) = 0$$

$$(2\Delta_3 + 1)(\Delta_{12} - A_{12}) + (\Delta_{13} - A_{13})(\Delta_{23} + A_{23}) = 0$$

$$(2\Delta_3 + 1)(A_{12} + \Delta_{12}) + (\Delta_{23} - A_{23})(\Delta_{13} + A_{13}) = 0 \qquad (4.16)$$

The state mentioned above happens to be also degenerate at level $1/2$ and relative charge -1, the null vector being

$$|\chi_{1/2}^- > = \bar{G}_{-1/2}|\Delta, Q > . \qquad (4.17)$$

In the same way we derive another equation:

$$\sum_{i=1}^{n} \left[\frac{\partial}{\partial \theta_i} - \bar{\theta}_i \frac{\partial}{\partial \theta_i} \right] < 0|\Phi_1(\mathbf{z}_1) \ldots \Phi_n(\mathbf{z}_n)\Phi(0)|0 > = 0 \qquad (4.18)$$

which for the three-point function in particular implies

$$A_{13} = -\Delta_{13}, A_{23} = -\Delta_{23}. \qquad (4.19)$$

Solving (4.16) and (4.19) we obtain

$$2\Delta_1 = Q_1, 2\Delta_2 = Q_2, \Delta_1 = \Delta_3 - \Delta_2 \qquad (4.20)$$

. Consequently in the operator product of Φ_2, with $2\Delta_2 = Q_2$, and Φ_3, only fields with $2\Delta_1 = Q_1$, and $\Delta_1 = \Delta_3 - \Delta_2$ can appear.

The superdifferential equations for the three-point functions are solved in Appendix B. Here we will present the "fusion" rules for the NS sector of degenerate theories with $\tilde{c} < 1$.

As was mentioned in the previous section, the unitary irreducible representations in the NS sector with $\tilde{c} < 1$, exist when $\tilde{c} = 1 - \frac{2}{m}, m\epsilon\mathbf{Z}^+ - \{1\}$ and their dimensions and changes are given by (3.26). It can be shown that for the family (j, k), there are three independent null hwv embedded in it, one at relative change zero and level $m - (j + k)$, another at relative change 1 and level k and another one at relative change -1 and level j, (for more details see ref. [12]). Consequently the correlation functions of (j, k) satisfy three superdifferential equations of orders $j, k, m - (j + k)$, simultaneously. The existence of three null vectors in the N=2 case is qualitatively different from the already N=0,1 cases.

The "fusion" rules coming from the consideration of the two charged null vectors at levels j_0, k_0 of the family (j_0, k_0) are the following:[*]

$$(j_0, k_0) \otimes (j, k) = \sum_{n=0}^{j_0 - k_0} (j + j_0 - k_0 - n, k - n), j_0 > k_0 \qquad (4.21a)$$

$$(j_0, k_0) \otimes (j, k) = \sum_{n=0}^{j_0 - k_0} (j - n, k - j_0 + k_0 - n), j_0 < k_0 \qquad (4.21b)$$

$$(j_0, j_0) \otimes (j, k) = \sum_{n=-j_0+1/2}^{j_0 - 1/2} (j + n, k + n) \qquad (4.21c)$$

As was mentioned above, the family (j_0, k_0) is also degenerate at relative charge zero and level $m - (j_0 + k_0)$. The extra conditions from this new null hwv have the effect of truncating the sums in (4.21) into the "unitary bounds", $0 < j', k', j' + k' \leq m - 1$, where $(j_0, k_0) \otimes (j, k) \sim (j', k')$. This truncation phenomenon is known already to happen in the analogous minimal theories of the $N = 0, 1$ algebras. Thus it is consistent to built $N = 2$ unitary minimal systems, with

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[*] In appendix C we present another heuristic justification of the fusion rules based on the unitary construction of these representations using free fermions.

$\tilde{c} < 1$, where there is a finite number of representations, all degenerate, and all the correlation functions calculable.

We present the two explicit examples of the operator algebra of the first two non-trivial theories with $\tilde{c} = 1/3, (m = 3), \tilde{c} = 1/2, (m = 2)$. In the $\tilde{c} = 1/3$ theory the representation content of the NS sector is shown in Fig. 1. The operator algebra is the following:

$$(\frac{1}{6}, \pm\frac{1}{3}) \otimes (0,0) \sim (0,0), \qquad (\frac{1}{6}, \pm\frac{1}{3}) \otimes (\frac{1}{6}, \mp\frac{1}{3}) \sim (0,0). \qquad (4.22)$$

This system, is somewhat special and it will be analyzed in more detail in the next section.

The $\tilde{c} = 1/2$ systems has the representation content shown in Fig.2. It's fusion rules are:

$$\begin{array}{ll}
(\frac{1}{8}, \pm\frac{1}{4}) \otimes (\frac{1}{8}, \mp\frac{1}{4}) \sim (0,0) & , (\frac{1}{8}, \pm\frac{1}{4}) \otimes (\frac{1}{4}, \mp\frac{1}{2}) \sim (\frac{1}{8}, \mp\frac{1}{4}) \\
(\frac{1}{8}, \pm\frac{1}{4}) \otimes (\frac{1}{2}, 0) \sim (\frac{1}{8}, \pm\frac{1}{4}) & , (\frac{1}{4}, \pm\frac{1}{2}) \otimes (\frac{1}{8}, \mp\frac{1}{4}) \sim (\frac{1}{8}, \mp\frac{1}{4}) \\
(\frac{1}{4}, \pm\frac{1}{2}) \otimes (\frac{1}{2}, 0) \sim (\frac{1}{4}, \pm\frac{1}{2}) & , (\frac{1}{2}, 0) \otimes (\frac{1}{2}, 0) \sim (\frac{1}{2}, 0) \oplus (0,0)
\end{array} \qquad (4.23)$$

We should remind the reader that the "fusion" rules we have derived, give the maximum possible set of operators that can appear in an operator product expansion. To determine exactly which of them contribute and to evaluate their Glebsch-Gordon coefficients one has to evaluate the 4-point function. This is what we will do for the $\tilde{c} = 1/3$ system in the next section.

5. The $\tilde{c} = \frac{1}{3}, N = 2$ Superconformal Theory

This theory has the simplest operator content compared to the other unitary minimal $N = 2$ theories. It is also the only member of the $N = 2$ discrete series which has the same central element with a member of the $N = 1$ discrete series. The authors of [10], identified some operators in the NS and R sector of the $N = 2, \tilde{c} = 1/3$ system with corresponding operators in the $\hat{c} = 2/3, N = 1$ system. In [12], the rest of the operators of the $N = 2$ system were identified and a rigorous proof of the decomposition was given using character formulae. The correspondence is as follows: (subscripts indicate $N = 1, 2$). The unit operator $(0)_2$ decomposes

into $(0)_1$ and a primary field $(1)_1$ in the NS sector. The energy momentum tensor and one of the supercharges are contained in $(0)_1$, where as the other supercharge and the $U(1)$ current are contained in $(1)_1$. The NS representation $(h,q) = (\frac{1}{6}, \pm\frac{1}{3})$ decomposes into $(\frac{1}{6})_1$ in the NS sector. In the R sector, $(\frac{3}{8}, 0)_2$ decomposes into $(\frac{3}{8})_1$ in the R sector of the $N=1$ system whereas $(\frac{1}{24}, \pm\frac{1}{3}, \pm\frac{2}{3})$ decomposes into $(\frac{1}{24})_1$ in the R sector. Finally $(\frac{1}{16})_2$ in the twisted sector of the $N=2$ theory decomposes into $(\frac{1}{16})_1$ in the NS sector of the $N=1$ theory. Thus the $\tilde{c} = 1/3$, $N=2$ system is a subsector of the $\hat{c} = 2/3$, $N=1$ system, as is shown in Fig. 3.

The general discussion of the previous section can be specialized in this situation. The model contains the unit (superfield) operator and a conjugate pair of primary operators, representing the $\Delta = \frac{1}{6}$, $Q = \pm\frac{1}{3}$ states of the model. We will denote by $\Phi_\pm$ and Φ_0 the corresponding superfield operators. The two point function is:

$$\langle 0 | \Phi_+(\mathbf{z}_1) \Phi_-(\mathbf{z}_2) | 0 \rangle = \mathbf{z}_{12}^{-1/3} \exp \left\{ -\frac{1}{3} \frac{\theta_{12}\bar{\theta}_{12}}{\mathbf{z}_{12}} \right\} \tag{5.1}$$

where we suppressed the antianalytic part and we've chosen a particular convenient normalization for the two-point function. The only three-point function which is non-zero is, $\langle 0 | \Phi_0(\mathbf{z}_1) \Phi_+(\mathbf{z}_2) \Phi_-(\mathbf{z}_3)) | 0 \rangle$. It is fixed up to a normalization constant by the $OSP(2|2)$ invariance and the extra differential equations that it is satisfying due to the fact that it contains degenerate fields.

$$\langle 0 | \Phi_0(\mathbf{z}_1) \Phi_+(\mathbf{z}_2) \Phi_-(\mathbf{z}_3) | 0 \rangle = C \mathbf{z}_{23}^{-1/3} \exp \left\{ -\frac{1}{3} \frac{\theta_{23}\bar{\theta}_{23}}{\mathbf{z}_{23}} \right\} \tag{5.2}$$

It implies the following operator product expansions for the component fields $\Phi_\pm(\mathbf{z}) \equiv \phi_\pm(z) + \theta\bar{\psi}_\pm(z) + \bar{\theta}\psi_\pm(z) + \theta\bar{\theta}g_\pm(z)$

$$\phi_+\phi_- \sim J, \quad \phi_+g_- \sim -\frac{1}{3}J, \quad \phi_-g_+ \sim -\frac{1}{3}J$$

$$\bar{\psi}_+\psi_- \sim \frac{1}{3}J, \quad \psi_+\bar{\psi}_- \sim \frac{1}{3}J, \quad g_+g_- \sim \frac{4}{9}J$$

which are determined up to an overall normalization constant. The first non-trivial correlation

function is the 4-point function. Its evaluation enables us to fix the Glebsch-Gordon coefficient in the OPE, in (5.3).

There are two ways to evaluate the 4-point function. One is to solve the superdifferential equations that it satisfies due to degeneracy of the operators contained in it. The other is to use the Feigin-Fuks construction. The only non-trivial 4-point function is $\langle 0|\Phi_-(\mathbf{z}_1)\Phi_+(\mathbf{z}_2)\Phi_-(\mathbf{z}_3)\Phi_+(\mathbf{z}_4)|0\rangle$. The operator $\Phi_+(\mathbf{z})$ is degenerate at level 1, relative change zero, at level $1/2$, relative charge one and at level $\frac{3}{2}$, relative change -1. The relevant superdifferential equation for the 4-point function $F_4(\tilde{\mathbf{z}}_1, \tilde{\mathbf{z}}_2, \tilde{\mathbf{z}}_3)$ are:

$$\left[\sum_{i=1}^{3} G_{1/2}^{i}\right] F_4(\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3) = 0 \tag{5.4a}$$

$$\sum_{i=1}^{3} \left[L_1^i - J_1^i\right] F_4(\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3) = 0 \tag{5.4b}$$

$$\left[\sum_{i=1}^{3} \bar{G}_{3/2}^{i} - \sum_{i,j=1}^{3} (J_1^i + L_1^i)\bar{G}_{1/2}^{j}\right] F_4(\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3) = 0 \tag{5.4c}$$

where the relevant differential operators are defined in (B.9) in appendix B, and we have simplified (5.4b) using (5.4a). The variables $\mathbf{z}_i$ are the shifted variables we mentioned at the last section.

Global $N = 2$ superconformal invariance constrains the four-point function to be of the form:

$$F_4(\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_4) = C(\mathbf{z}_{12}\mathbf{z}_{34})^{-1/3} \exp\left[\frac{1}{3}\left(\frac{\theta_{14}\bar{\theta}_{14}}{\mathbf{z}_{14}} - \frac{\theta_{24}\bar{\theta}_{24}}{\mathbf{z}_{24}} + \frac{\theta_{34}\bar{\theta}_{34}}{\mathbf{z}_{34}}\right)\right] \times$$

$$\times G_4(x_1, x_2, x_3, x_4) \tag{5.5}$$

where $x_1, i = 1, 2, 3, 4$ are the four independent combinations of the coordinates invariant under

the OSP(2|2) group. The obvious (dependent) invariants are:

$$x_1 = \frac{\theta_{23}\bar{\theta}_{23}}{\mathbf{z}_{23}} + \frac{\theta_{34}\bar{\theta}_{34}}{\mathbf{z}_{34}} - \frac{\theta_{24}\bar{\theta}_{24}}{\mathbf{z}_{24}}$$

$$x_2 = \frac{\theta_{12}\bar{\theta}_{12}}{\mathbf{z}_{12}} + \frac{\theta_{24}\bar{\theta}_{24}}{\mathbf{z}_{24}} - \frac{\theta_{14}\bar{\theta}_{14}}{\mathbf{z}_{14}}$$

$$x_3 = \frac{\theta_{13}\bar{\theta}_{13}}{\mathbf{z}_{13}} + \frac{\theta_{34}\bar{\theta}_{34}}{\mathbf{z}_{34}} - \frac{\theta_{14}\bar{\theta}_{14}}{\mathbf{z}_{14}}$$

$$y_1 = \frac{\theta_{12}\bar{\theta}_{12}}{\mathbf{z}_{12}} + \frac{\theta_{23}\bar{\theta}_{23}}{\mathbf{z}_{23}} - \frac{\theta_{13}\bar{\theta}_{13}}{\mathbf{z}_{13}}$$

$$y_2 = \frac{\mathbf{z}_{14}\mathbf{z}_{23}}{\mathbf{z}_{12}\mathbf{z}_{34}}, \quad y_3 = \frac{\mathbf{z}_{13}\mathbf{z}_{24}}{\mathbf{z}_{12}\mathbf{z}_{34}} \tag{5.6}$$

Since $y_1 = x_1 + x_2 - x_3$, y_1 can be deleted. We have also the additional relations:

$$x_1^2 = x_2^2 = x_3^2 = y_1^2 = 0, \quad x_1 x_2 = (x_1 + x_3)x_2, \quad x_1 x_2 x_3 = 0 \tag{5.7a}$$

$$(y_2 - y_3 + 1)^2 = 2y_2 x_1 x_2, \quad x_2 x_3 = y_2 x_1 x_3, \quad x_1 x_2 = y_3 x_1 x_3 \tag{5.7b}$$

The relations above imply that in fact x_1, x_2, x_3 and $x_4 \equiv y_2$ are independent invariants. Solving eqs. (5.4) we arrive at a four-point function of the form:

$$G_4(x_1, x_2, x_3, x_4) = C \left(\frac{x_4 + 1}{x_4} \right)^{1/3} \exp \left[\frac{1}{3(x_4 + 1)} (y - x_1 + x_4 x_2) \right] \tag{5.8}$$

where $y \equiv y_2 - y_3 + 1$.

The four-point function, (5.8), is powerlike, something to be expected since the primary fields of the $\tilde{c} = 1/3$ theory can be constructed as vertex operators of a single $c = 1$ scalar field (see app. D). We have performed the same calculations using the vertex operator method, [17, 21]. We find the same result as in (5.8)[*] It is difficult though in this method to obtain the result as a super meromorphic function in $N = 2$ superspace.

---

[*] The hypergeometric function obtained through this method truncate to polynomials of the first degree giving a powerlike four-point function.

By factorizing over two-point functions we can find that $C = 1$. This implies that the OPE coefficient in (5.3) is in fact unity.

The full construction of the four-point function, including its anti-holomorphic part does not involve any subleties related to monodromy invariance, (locality in the Euclidean domain). We simply have to multiply the holomorphic and antiholomorphic pieces which have the same form. Knowledge of the four-point function (5.8) is enough to determine any $n-$point function using the OPE coefficient for the degenerate operators.

Closing this section we shall remark again that the $\tilde{c} = 1/3$, $N = 2$ superconformal theory describes a particular point of the Gaussian model for a specific value of the radius (see [15] and app. D).

6. Conclusions and Prospects

In this paper we analyzed the general structure of $N = 2$ superconformal invariant theories. We developed the local analytic geometry of $(2,0)$ complex superspace and we constructed the global $N = 2$ superconformal group. The Ward identifies for the global $N = 2$ superconformal symmetry were solved, which provided a partial determination of the correlation functions. In particular the two-point functions are determined up to an irrelevant normalization, whereas the three-point function is determined up to two OPE coefficients.

We then specialized to unitary minimal theories with $\tilde{c} < 1$. We derived extra superdifferential equations, satisfied by all the correlation functions of the degenerate operators. Solving these equations for the three-point function we derived the "fusion" rules for the $N = 2$ unitary minimal systems. In particular we showed that the operator algebra of the unitary degenerate representations with $\tilde{c} < 1$ closes, which in turn guarantees the consistency of $N = 2$ superconformal unitary minimal theories with $\tilde{c} < 1$. We analyzed in particular the simplest such system, that is the one with $\tilde{c} = 1/3$. This system has been shown to be a subsector of the $\hat{c} = 2/3, N = 1$ superconformal minimal system, [10, 12]. We calculated its four-point function by solving the relevant superdifferential equation and we thus determined its operator algebra.

The $\tilde{c} = 1/3$ system is realized at some special point in the Gaussian model, [15]. We think that it is interesting and important to search for critical systems which realize the $N = 2$ superconformal symmetry since their structure seems to be very exciting.

Of course much more needs to be done in the context of $N = 2$ superconformally invariant theories. A unitary construction of the degenerate representations with $\tilde{c} \geq 1$ is still missing. Their operator algebra needs to be found. (Work in that direction in in progress).

This class of representations is very important since they occur in $N = 2$ non-linear σ-models on compact Ricci-flat manifolds, arising in superstring compactification. The existence of four-dimensional supersymmetry relies heavily on the $N = 2$ superconformal invariance of the respective σ-model, [14].

$N = 2$ superconformal methods may turn out to be important tools in understanding superstring compactification and low energy superstring phenomenology.

Note added.

During the completion of this work we received ref [28] where the analytic geometry of extended super Rieman surfaces was developed. We do agree with the results of ref [24] concerning the local geometry (since we haven't dealt with global aspects).

Appendix A

In this appendix we discuss the derivation of the two point functions in the $R^{\pm}$ and T sectors of an $N = 2$ superconformal theory. They can be derived as in the NS sector by solving the appropriate Ward identities. But since the R and T sectors, this procedure is somewhat complicated, we will use another device, which proved to be useful in a lot of situations in the past, [23]. This is the realization of the $N = 2$ algebra and it's hw irreducible representations in an appropriate extended "Coulomb gas" system. The primary superfields are realized as vertex operators of the free fields of the theory. The building blocks are two chiral $N = 2$ scalar free superfields $X, \bar{X}$. The $N = 2$ algebra is realized in the usual way, and $\tilde{c} \neq 1$ is obtained by putting non-trivial boundary conditions at infinity. The vertex operator method for the $N = 2$ algebra has been developed in [17, 21].

Thus we have two chiral scalar superfields:

$$\bar{D}X = D\bar{X} = 0 \qquad (A.1)$$

with free propagators:

$$\langle 0 | X(\mathbf{z}_1)\bar{X}(\mathbf{z}_2) | 0 \rangle = -ln(\mathbf{z}_{12}) - \frac{1}{2}\frac{\theta_{12}\bar{\theta}_{12}}{\mathbf{z}_{12}}$$

$$\langle 0 | \bar{X}(\mathbf{z}_1)X(\mathbf{z}_2) | 0 \rangle = -ln(\mathbf{z}_{12}) + \frac{1}{2}\frac{\theta_{12}\bar{\theta}_{12}}{\mathbf{z}_{12}}$$

$$\langle 0 | \bar{X}(\mathbf{z}_1)\bar{X}(\mathbf{z}_2) | 0 \rangle = \langle 0 | X(\mathbf{z}_1)X(\mathbf{z}_2) | 0 \rangle = 0 \qquad (A.2)$$

and the primary fields of the theory are represented by vertex operators:

$$V\alpha, \bar{\alpha}(\mathbf{z}) \equiv : \exp[i\alpha\bar{X}(\mathbf{z}) + i\bar{\alpha}X(\mathbf{z})] : \qquad (A.3)$$

with dimension $\Delta_{\alpha,\bar{\alpha}} = \alpha\bar{\alpha} + \frac{1}{2}(\bar{\alpha}\beta + \alpha\bar{\beta})$, and charge $Q_{\alpha\bar{\alpha}} = (\bar{\alpha}\beta - \alpha\bar{\beta})$ where $\beta, \bar{\beta}$ are the charges at infinity modifying the standard boundary conditions of the free system. The central

charge of the $N = 2$ algebra is given by: $\tilde{c} = 1 - 2\beta\bar{\beta}$. Correlation functions of the vertex operators are easy to obtain since $X, \bar{X}$ are free fields.

$$\langle 0|V_{\alpha_1,\bar{\alpha}_1}(\mathbf{z}_1)\ldots V_{\alpha_n,\bar{\alpha}_n}(\mathbf{z}_n)|0\rangle = \prod_{i<j}^{n} \mathbf{z}_{ij}^{(\alpha_i\bar{\alpha}_j+\bar{\alpha}_i\alpha_j)} \times$$

$$\times \exp\left[\sum_{i<j}^{n} -\frac{1}{2}(\alpha_i\bar{\alpha}_j - \bar{\alpha}_i\alpha_j)\frac{\theta_{ij}\bar{\theta}_{ij}}{\mathbf{z}_{ij}}\right] \delta(\sum_i \alpha_i + \beta)\delta(\sum_i \bar{\alpha}_i + \bar{\beta}) \qquad (A.4)$$

where the delta functions impose the "change neutrality" to the system, which is essential in order for the IR divergences to cancel, otherwise they force the correlation functions to vanish, (see [23]). For example in the NS sector the two-point function is:

$$\langle 0|V_{\alpha_1,\bar{\alpha}_1,}(\mathbf{z}_1)V_{\alpha_2,\bar{\alpha}_2}(\mathbf{z}_2)|0\rangle = \mathbf{z}_{12}^{\alpha_1\bar{\alpha}_2+\bar{\alpha}_1\alpha_2} \exp\left\{-\frac{1}{2}(\alpha_1\bar{\alpha}_2 - \bar{\alpha}_1\alpha_2)\frac{\theta_{12}\bar{\theta}_{12}}{\mathbf{z}_{12}}\right\}$$

$$\times\delta(\alpha_1 + \alpha_2 + \beta) \times \delta(\bar{\alpha}_1 + \bar{\alpha}_2 + \bar{\beta}) = \mathbf{z}_{12}^{-(\Delta_1+\Delta_2)} \exp\left\{\frac{Q_2\theta_{12}\bar{\theta}_{12}}{\mathbf{z}_{12}}\right\}\delta_{\Delta_1,\Delta_2}, \delta_{Q_1,-Q_2} \qquad (A.5)$$

the same that has been obtained in section 4.

In the R sector one has to impose periodic boundary conditions for the fermionic components of $X, \bar{X}$, giving rise to the modified free propagators:

$$\langle 0|X(\mathbf{z}_1)\bar{X}(\mathbf{z}_2)|0\rangle_R = \langle 0|X(\mathbf{z}_1)\bar{X}(\mathbf{z}_2)|0\rangle_{NS} + \theta_1\bar{\theta}_2\left\{\frac{(\sqrt{z_1} - \sqrt{z_2})^2}{2\sqrt{z_1 z_2}(z_1 - z_2)}\right\} \qquad (A.6a)$$

$$\langle 0|\bar{X}(\mathbf{z}_1)X(\mathbf{z}_2)|0\rangle_R = \langle 0|\bar{X}(\mathbf{z}_1)X(\mathbf{z}_2)|0\rangle_{NS} + \bar{\theta}_1\theta_2\left\{\frac{(\sqrt{z_1} - \sqrt{z_2})^2}{2\sqrt{z_1 z_2}(z_1 - z_2)}\right\} \qquad (A.6b)$$

Also L_0 is shifted in this sector by $1/8$ and J_0 by $\pm\frac{1}{2}$ (due to the zero modes of the fermionic components), so that the vertex operations in the R sector have dimensions $\Delta = \alpha\bar{\alpha} + \frac{1}{2}(\bar{\alpha}\beta + \alpha\bar{\beta}) + \frac{1}{8}$ and charge $Q = \bar{\alpha}\beta - \alpha\bar{\beta} \pm \frac{1}{2}$.

The two-point function in the R sector can be obtained by directly applying (A.4). In component form it is given by:

$$\langle 0|\Phi_+(\mathbf{z}_1)\Phi_-(\mathbf{z}_2)|0\rangle_R = \frac{1}{Z_{12}^{2\Delta}} - Q\frac{\theta_1\bar{\theta}_1 + \theta_2\bar{\theta}_2}{z_{12}^{2\Delta+1}} + \Delta\frac{(\sqrt{z_1}+\sqrt{z_2})^2}{2\sqrt{z_1 z_2}z_{12}^{2\Delta+1}}(\theta_1\bar{\theta}_2 + \bar{\theta}_1\theta_2)$$

$$+Q\frac{\theta_1\bar{\theta}_2 - \bar{\theta}_1\theta_2}{z_{12}^{2\Delta+1}} + \left[\Delta^2\frac{(\sqrt{z_1}+\sqrt{z_2})^2}{4z_1 z_2} - Q^2\frac{(\sqrt{z_1}+\sqrt{z_2})^4}{4z_1 z_2 z_{12}} + 2\Delta\right]\frac{\theta_1\bar{\theta}_1\theta_2\bar{\theta}_2}{z_{12}^{2\Delta+2}} \qquad (A.7)$$

where $\Delta_1 = \Delta_2$, $\quad \Delta = \Delta_1 - 1/8$, $\quad Q_1 + Q_2 = 0$, $\quad Q = Q_1 - 1/2$.

In the T-sector one needs to twist one of the scalars in X and $\bar{X}$ as well as one of the fermions. The free propagator becomes:

$$\langle 0|[X(\mathbf{z}_1) + \bar{X}(\mathbf{z}_1)][X(\mathbf{z}_2) + \bar{X}(\mathbf{z}_2)]|0\rangle = -2ln(\mathbf{z}_{12})$$
$$- \theta_1^1\theta_2^1\frac{z_1 + z_2}{2\sqrt{z_1 z_2}} - \frac{\theta_1^2\theta_2^2}{z_{12}} + \theta_1^1\theta_1^2\theta_2^1\theta_2^2\frac{z_1 + z_2}{\sqrt{z_1 z_2}z_{12}^2} \qquad (A.8)$$

where we used $O(2)$ superspace variables, (the upper index is the $O(2)$ index).

Consistency for the vertex operators construction implies $\alpha = \bar{\alpha}, \beta = \bar{\beta}$ in this case. Also L_0 is shifted by $1/8$ as in the R sector. The two-point function is thus given in component form by:

$$\langle 0|\Phi_1(\mathbf{z}_1)\Phi_2(\mathbf{z}_2)|0\rangle = \frac{1}{\mathbf{z}_{12}^{2\Delta}}\left\{1 + \Delta\frac{z_1 + z_2}{2\sqrt{z_1 z_2}z_{12}}\delta^{ij}\theta_1^i\theta_2^j\right.$$
$$\left. - \Delta(\Delta+1)\frac{z_1 + z_2}{2\sqrt{z_1 z_2}z_{12}^2}\epsilon^{ij}\epsilon^{kl}\theta_1^i\theta_1^j\theta_2^k\theta_2^l\right\} \qquad (A.9)$$

where again $\Delta_1 = \Delta_2$, $\quad \Delta = \Delta_1, -1/8$.

Using the procedure above it is straightforward to evaluate the general form of any correlation function in R and T-sector. In some cases screening operators are needed, see [17, 22] for details.

In this appendix we solve the first few superdifferential equations for the three-point function and derive the conditions leading to the "fusion" rules discussed in section 4.

For the representation (Δ_3, Q_3), degenerate at level $1/2$ and relative change 1 the null *hwv* is:

$$|\chi_{1/2}^+\rangle = G_{-1/2}|\Delta_3, Q_3\rangle \qquad (B.1)$$

It implies the following equation for the three-point function

$$\sum_{i=1}^{2}\Big[\frac{\partial}{\partial\bar{\theta}_i} - \theta_i\frac{\partial}{\partial z_i}\Big]\langle 0|\Phi_1(\mathbf{z}_1)\Phi_2(\mathbf{z}_2)\Phi_3(0)|0\rangle = 0 \qquad (B.2)$$

Substituting the general form of the three-point function in (B.2) we obtain:

$$A_{13} = \Delta_{13}, \quad A_{23} = \Delta_{23} \qquad (B.3)$$

Before we continue, it is convenient to introduce some notations concerning the superdifferential operations we use. We define:

$$
\begin{aligned}
\hat{L}_n^i &= z_i^{1-n}\frac{\partial}{\partial z_i} - (n-1)z_i^{-n}\Big[\Delta_i + \frac{1}{2}(\theta_i\frac{\partial}{\partial\theta_i} + \bar{\theta}_i\frac{\partial}{\partial\theta_i}\Big] + \frac{Q_i}{2}n(n-1)z_i^{-n-1}\theta_i\bar{\theta}_i \\
\hat{J}_n^i &= z_i^{-n}\Big[Q_i + \bar{\theta}_i\frac{\partial}{\partial\theta_i} - \theta_i\frac{\partial}{\partial\theta_i} - 2nz_i^{-1}\Delta_i\theta_i\bar{\theta}_i\Big] \\
\hat{G}_r^i &= z_i^{\frac{1}{2}-r}\Big[\frac{\partial}{\partial\bar{\theta}_i} - \theta_i\frac{\partial}{\partial z_i}\Big] + (r-1/2)(2\Delta_i+q_i)z_i^{-r-1/2}\theta_i + (r-1/2)z_i^{-r-1/2}\theta_i\bar{\theta}_i\frac{\partial}{\partial\theta_i} \\
\hat{\bar{G}}_r^i &= z_i^{\frac{1}{2}-r}\Big[\frac{\partial}{\partial\theta_i} - \bar{\theta}_i\frac{\partial}{\partial z_i}\Big] + (r-1/2)(2\Delta_i-Q_i)z_i^{-r-1/2}\bar{\theta}_i + (r-1/2)z_i^{-r-1/2}\theta_i\bar{\theta}_i\frac{\partial}{\partial\bar{\theta}_i}
\end{aligned}
\qquad (B.4)
$$

The conditions coming from the null *hwv* at level one and relative change zero have been discussed in the main body of the paper. At level $3/2$ and relative change 1 the null *hwv* is

$$|\chi_{3/2}^+\rangle = \Big[(\Delta_3 - \frac{Q_3}{2})G_{-3/2} + (J_{-1} - L_{-1})G_{-1/2}\Big]|\Delta_3, Q_3\rangle \qquad (B.5)$$

implying the following equations for the three point function

$$\left[\left(\Delta_3 - \frac{Q_3}{2}\right)\sum_{i=1}^{2}\hat{G}^i_{3/2} + \sum_{i=1}^{2}\sum_{j=1}^{2}(\hat{J}^i_1 - \hat{L}^i_1)\hat{G}^j_{1/2}\right]\langle 0|\Phi_1(\mathbf{z}_1)\Phi_2(\mathbf{z}_2)\Phi_3(0)|0\rangle = 0 \qquad (B.6)$$

which give after substituting the three-point function in:

$$(2\Delta_3 - Q_3)(\Delta_{12} - A_{12}) + 2(Q_2 + \Delta_{23})(\Delta_{13} - A_{13}) = 0 \qquad (B.7)$$

Finally at level $5/2$ and relative change 1 the differential equation is:

$$\left[(2\Delta_3 - Q_3 + 4)(2\Delta_3 - 2Q_3 + 3)\sum_{i=1}^{2}\hat{G}^i_{5/2} - 3(2\Delta_3 - Q_3 + 4)\sum_{i=1}^{2}\sum_{j=1}^{2}\hat{G}^i_{3/2}(2\hat{J}^j_1 - 2\hat{L}^j_1)\right.$$

$$\left.+\hat{\Lambda}\right]\langle 0|\Phi_1(\mathbf{z}_1)\Phi_2(\mathbf{z}_2)\Phi_3(0)|0\rangle = 0 \qquad (B.8)$$

where

$$\hat{\Lambda} \equiv \sum_{i=1}^{2}\hat{G}^i_{1/2}\sum_{j=1}^{2}\left[(2\Delta_3 - 2Q_3 + 3)(2\hat{L}^j_2 - 3\hat{J}^j_2) + 12\hat{J}^j_2 + \sum_{k=1}^{2}\left\{6\hat{L}^j_1\hat{L}^k_1\right.\right.$$

$$\left.\left.+12\hat{J}^j_1\hat{J}^k_1 - 18\hat{J}^j_1\hat{L}^k_1 + 3G^j_{3/2}\bar{G}^k_{1/2}\right\}\right] \qquad (B.8')$$

implying the following set of conditions

$$(\Delta_{13} - A_{13})[(2\Delta_3 - Q_3)(2\Delta_3 - 2Q_3 - 3) - 3(\Delta_{13} + 1)(2\Delta_3 - Q_3 - 2)$$

$$- 6(Q_1 - 1)(2\Delta_3 - Q_3) - 3(Q_1 - 1)(2\Delta_3 - 2Q_3 - 1)$$

$$- (2\Delta_3 - 2Q_3 + 3)(2\Delta_{13} + 2\Delta_1 + 3) + 6(\Delta_{13} + 1)(\Delta_{13} + 2)$$

$$+ 18(Q_1 - 1)(\Delta_{13} + 1) + 12(Q_1 - 1)^2] + (2\Delta_1 + Q_1)$$

$$\times [2(2\Delta_3 - Q_3)(2\Delta_3 - 2Q_3 - 3) - 3\Delta_{13}(2\Delta_3 - Q_3 - 2) - 6(Q_1 - 1)$$

$$\times (2\Delta_3 - Q_3) - 3(\Delta_{13} + A_{13})] = 0$$

$(B.9a)$

$$(\Delta_{12} - A_{12})[-(2\Delta_3 - Q_3)(2\Delta_3 - 2Q_3 - 3) + 3\Delta_{13}(2\Delta_3 - Q_3 - 2)$$
$$+ 6(Q_1 - 1)(2\Delta_3 - Q_3) + 3(\Delta_{13} + A_{13})] + (\Delta_{13} - A_{13})$$
$$\times [-3(2\Delta_3 - Q_3 - 2)\Delta_{23} - 6Q_2(2\Delta_3 - Q_3) + 2\Delta_{12}(2\Delta_3 - 2Q_3 + 3)$$
$$- 3(\Delta_{23} + A_{23}) + 12\Delta_{23}(\Delta_{13} + 1) + 18Q_2(\Delta_{13} + 1) + 18(Q_1 - 1)$$
$$\times \Delta_{23} + 24(Q_1 - 1)Q_2] + (2\Delta_1 + Q_1)[-3\Delta_{23}(2\Delta_3 - Q_3 - 2)$$
$$- 6Q_2(2\Delta_3 - Q_3) - 3(\Delta_{23} + A_{23})] = 0$$

$$(B.9b)$$

$$(\Delta_{12} - A_{12})[-(2\Delta_3 - Q_3)(2\Delta_3 - 2Q_3 - 3) + 3\Delta_{23}(2\Delta_3 - Q_3 - 2) + 6Q_2$$
$$\times (2\Delta_3 - Q_3) + 3(\Delta_{23} + A_{23})] + (\Delta_{13} - A_{13})[-3Q_2$$
$$(2\Delta_3 - 2Q_3 - 1) - 2(2\Delta_3 - 2Q_3 + 3)(\Delta_2 + \Delta_{23}) + 6\Delta_{23}(\Delta_{23} + 1)$$
$$+ 18Q_2\Delta_{23} + 12Q_2^2 + 3(A_{23} + \Delta_{23})] = 0$$

$$(B.9c)$$

$$(\Delta_{12} + A_{12})[-(2\Delta_3 - Q_3)(2\Delta_3 - 2Q_3 - 3) + 3\Delta_{13}(2\Delta_3 - Q_3 - 2)$$
$$+ 6Q_1(2\Delta_3 - Q_3) + 3(\Delta_{13} + A_{13})] + (\Delta_{23} - A_{23})[-3Q_1$$
$$(2\Delta_3 - 2Q_3 - 1) - 2(2\Delta_3 - 2Q_3 + 3) \times (\Delta_1 + \Delta_{13})$$
$$+ 6\Delta_{13}(\Delta_{13} + 1) + 18Q_1\Delta_{13} + 12Q_1^2 + 3(\Delta_{13} + A_{13})] = 0$$

$$(B.9d)$$

$$(\Delta_{23} - A_{23})[(2\Delta_3 - Q_3)(2\Delta_3 - 2Q_3 - 3) - 3(2\Delta_3 - Q_3 - 2)(\Delta_{23} + 1)$$
$$- 6(Q_2 - 1)(2\Delta_3Q_3) - 3(Q_2 - 1)(2\Delta_3 - 2Q_3 - 1)(2\Delta_3 - 2Q_3 + 3)$$
$$(2\Delta_{23} + 2\Delta_2 + 3) + 6(\Delta_{23} + 1)(\Delta_{23} + 2) + 18(Q_2 - 1)(\Delta_{23} + 1)$$
$$+ 12(Q_2 - 1)^2] + (2\Delta_2 + Q_2)[2(2\Delta_3 - Q_3)(2\Delta_3 - 2\Delta_3 - 3)$$
$$- 3\Delta_{23}(2\Delta_3 - Q_3 - 2) - 6(Q_2 - 1)(2\Delta_3 - Q_3)$$
$$- 3(\Delta_{23} + A_{23})] = 0$$

$$(B.9e)$$

$$(\Delta_{12} + A_{12})[-(2\Delta_3 - Q_3)(2\Delta_3 - 2Q_3 + 3) + 3\Delta_{23}(2\Delta_3 - Q_3 - 2) + 6(Q_2 - 1)$$

$$(2\Delta_1 - Q_3) + 3(\Delta_{23} + A_{23})] + (2\Delta_2 + Q_2)[-3\Delta_{13}(2\Delta_3 - Q_3 - 2)$$

$$- 6Q_1(2\Delta_3 - Q_3) - 3(\Delta_{13} + A_{13})] + (\Delta_{23} - A_{23})$$

$$[-3\Delta_{13}(2\Delta_3 - Q_3 - 2) - 6Q_1(2\Delta_3 - Q_3) + 2\Delta_{12}(2\Delta_3 - 2Q_3 + 3)$$

$$+ 12\Delta_{13}(\Delta_{23} + 1) + 18Q_1(\Delta_{23} + 1) + 18\Delta_{13}(Q_2 - 1)$$

$$+ 24Q_1(Q_2 - 1) - 3(\Delta_{13} + A_{13})] = 0$$

$$(B.9f)$$

The null hwv at level $n_0 \epsilon \mathbf{Z}^+ + \frac{1}{2}$ and relative change -1 are obtained from those with relative change 1 by making the following substitutions: $J_n \to -J_n, G_r \leftrightarrow \bar{G}_r$ and $Q \to -Q$. Consequently the conditions derived from the three-point function are those of relative change 1 with the additional substitution $Q_i \to -Q_i, A_{ij} \to -A_{ij}$.

Appendix C

In this appendix we present a heuristic justification of the "fusion rules" obtained for the NS-sector in section 4, and we extend it to derive also the fusion rules for the R- and T-sectors of the $N = 2$ algebra. We make use of the unitary construction of these representations using free fermions, [17].

In order to achieve this goal, n $SU(2)$ doublets with $U(1)$ charge zero and an $SU(2)$ singlet with $U(1)$ charge ± 1 are used. They are hw irreducible representation of the $SU(2) \times U(1)$ affine algebra. The central change of the $N = 2$ algebra constructed this way turns out to be $\tilde{c} = 1 - \frac{2}{n+2}$ and the $U(1)$ current is given by $J^3(z) - I(z)$, where $I(z)$ is the $U(1)$ generator of the $SU(2) \times U(1)$ affine algebra. The hwv are constructed out of the modes $\psi^i_{-1/2}$ of the fermions acting on the vacuum, and we can multiply at most n of them so that the $SU(2)$ isospin of the generated states can take the values $l = 0, \frac{1}{2}, 1, \ldots, \frac{n}{2}$ while the third component takes the usual values $-l \leq l_3 \leq l$

The dimensions and $U(1)$ changes of the corresponding hwv's are:

$$\Delta = \frac{l(l+1) - l_3^2}{n+2}, Q = \frac{2l_3}{n+2} \qquad (C.1)$$

The above are hw irreducible representations of the affine algebra and they are all integrable. Their operator product rules are the same as in the $SU(2)$ Lie algebra case:

$$(l_1, m_1) \otimes (l_2, m_2) = \sum_{l=|l_1-l_2|}^{l'} (l, m_1 + m_2) \qquad (C.2)$$

where $l' = \min(\frac{n}{2}, l_1 - l_2)$, and the upper truncation is due to the integrability requirements, see for example, [24].

Now in order to make contact with the parametrization used in section 3, 4, we set $m = n + 2$, $\quad j = l + l_3 + 1/2$, $\quad k = l - l_3 + 1/2$. Then (C.2) reduces to the "fusion rules" (4.21).

This "derivation" can be applied to the R-sector as well giving the same fusion rules as in the NS sector.

In the T-sector a similar construction can be made, [17], using twisted fermions generating a twisted $SU(2) \times U(1)$ affine algebra. Only J^2 has integer modes and the dimensions of the hwv of the $N = 2$ algebra are given by:

$$\Delta = \frac{\tilde{c}}{8} + \frac{j_2^2}{4(n+2)} \qquad (C.3)$$

in terms of the eigenvalues j_2 of J_2, $\quad \frac{n}{2} \le j_2 \le \frac{n}{2}$.

To make contact with our notation in section 3 we have to identify: $2j_2 = m - 2r$. The fusion rules for the T-sector then, are:

$$[r_1] \otimes [r_2] = [r] \qquad (C.4)$$

where $r = r_1 + r_2$, if $r_1 + r_2 \le \frac{m}{2}$ or $r = r_1 + r_2 - \frac{m}{2}$ if $r_1 + r_2 > \frac{m}{2}$.

There is no rigorous proof that the above constructions do indeed generate the hwv of the degenerate representations. In that sense the derivation above is heuristic.

Appendix D

In this appendix we construct the components of the primary superfields of the $\tilde{c} = 1/3$ $N = 2$ superconformal system (NS sector) using a single $c = 1$ scalar field. We use these operators to give an alternative calculation of the four-point function (5.8) which was computed in the main body of this paper.

We consider a scalar field $\phi(z)$ with a two-point function given by:

$$\langle \, 0 \, | \, \phi(z)\phi(w) \, | \, 0 \, \rangle = -ln(z-w) \qquad (D.1)$$

We define the standard energy momentum tensor $T(z) = -\frac{1}{2} : \partial_z\phi\partial_z\phi :$ satisfying:

$$T(z)T(w) = \frac{\frac{1}{2}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + \text{nonsingular terms} \qquad (D.2)$$

A vertex operator defined by $V_a(z) \equiv \, : e^{ia\phi(z)} :$ has dimension $\Delta_a = \frac{a^2}{2}$:

$$T(z)V_a(w) = \frac{a^2}{2}\frac{V_a(w)}{(z-w)^2} + \frac{\partial_w V_a(w)}{z-w} + \text{nonsingular}. \qquad (D.3)$$

In this system the $N = 2$ superconformal algebra is realized by $T(z)$ and, [15],

$$J(z) \equiv \frac{i}{\sqrt{3}}\partial_z\phi(z), \quad G(z) \equiv \sqrt{2/3} : e^{i\sqrt{3}\phi(z)} :, \quad \bar{G}(z) = \sqrt{2/3} : e^{-i\sqrt{3}\phi(z)} : \qquad (D.4)$$

We can evaluate operator product expansions of vertex operators using the familiar formula:

$$V_a(z)V_b(w) = (z-w)^{ab} : e^{ia\phi(z)+ib\phi(w)} : \qquad (D.5)$$

by expanding the second exponential around $z = w$ and keeping the singular terms. Since:

$$J(z)V_a(w) = \frac{a}{\sqrt{3}}\frac{V_a(w)}{z-w}$$

we can easily establish that $T(z)$, $G(z)$, $\bar{G}(z)$ and $J(z)$ satisfy the standard $N = 2$ superconformal algebra (2.14) or (2.25) with $\tilde{c} = 1/3$.

Candidates for the lowest components of the primary superfields $\phi_\pm(z)$ with dimension $1/6$ and charge $\pm 1/3$ are the vertex operators:

$$\phi_+(z) \equiv\ :e^{\frac{i}{\sqrt{3}}\phi(z)}:, \quad \phi_-(z) =:\ e^{-\frac{i}{\sqrt{3}}\phi(z)}: \tag{D.7}$$

which by (D.3) and (D.6) have the correct dimension and U(1) charge. We have now to find the superpartners of $\phi_\pm$. Using the relations (3.6) in component form we have that:

$$G(z)\phi_\pm(w) = \frac{\psi_\pm(w)}{z-w} + \text{nonsingular} \tag{D.8a}$$

$$\bar{G}(z)\phi_\pm(w) = \frac{\bar{\psi}_\pm(w)}{z-w} + \text{nonsingular.} \tag{D.8b}$$

Applying (D.8a,b) to (D.7) we find

$$\psi_+(z) = 0, \quad \bar{\psi}_+(z) = \sqrt{2/3}\ :e^{-\frac{2i}{\sqrt{3}}\phi(z)}:$$

$$\psi_-(z) = \sqrt{2/3}\ :e^{\frac{2i}{\sqrt{3}}\phi(z)}:\ \quad \bar{\psi}_-(z) = 0 \tag{D.9}$$

Using then:

$$G(z)\psi_\pm(w) = 0, \quad \bar{G}(z)\bar{\psi}_\pm(w) = 0$$

$$G(z)\bar{\psi}_\pm(w) = (2\Delta + Q)\frac{\phi_\pm(w)}{(z-w)^2} + \frac{\partial_w\phi_\pm(w)}{z-w} + \frac{g_\pm(w)}{z-w}$$

$$\bar{G}(z)\psi_\pm(w) = (2\Delta - Q)\frac{\phi_\pm(w)}{(z-w)^2} + \frac{\partial_w\phi_\pm(w)}{z-w} - \frac{g_\pm(w)}{z-w} \tag{D.10}$$

we find that they are satisfied if: $g_+(z) = \partial_z\phi_+(z)$ and $g_-(z) = -\partial_z\phi_-(z)$.

The fact that one of the fermionic components is zero and the fourth component is a descendant of the first component explains the group theoretic result, [12], that the family $(\Delta = \frac{1}{6},\ Q = \pm\frac{1}{3})$ decomposes to the $N = 1$ family with $\Delta = \frac{1}{6}$ and half the apparent degrees of freedom.

This means, using our definition (2.8a,b) that $\phi_\pm$ are chiral primary operators of opposite chirality. In fact, looking at (3.11) we can establish that any primary superfield, degenerate at $n_0 = \pm 1/2$, is chiral in the sense of (2.8a,b) and thus contains half the apparent degrees of freedom.

Computing correlation functions of Φ_+ and Φ_- is now trivial. Using:

$$\left\langle\, 0\,|\, V_{a_1}(z_1)V_{a_2}(z_2)\ldots V_{a_n}(z_n)\,|\,0\,\right\rangle = \prod_{i<j}^{n}(z_{ij})^{a_i a_j}\delta\left(\sum_i a_i\right) \tag{D.11}$$

We can evaluate the different components of (5.8). Such a correlation is non-zero only if $\sum_i a_i = 0$, otherwise IR divergences force it to vanish. Such a calculation has been performed for the four-point function and as expected it agrees with the result (5.8).

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Figure Captions

Fig.1 : Operator content of the $\tilde{c} = 1/3$ minimal system (NS-sector) (the first entry is the dimension of an operator the second its change).

Fig.2 : Operator content of the $\tilde{c} = 1/2$ minimal system (NS sector).

Fig.3 : Operator content of the $\hat{c} = 2/3, N = 1$ minimal system.

Squares represent operators belonging to the NS sector, whereas circles represent those in the R-sector. The full squares and circles indicates the $N = 2$ subsystem of $\tilde{c} = 1/3$.