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# GLOBAL GAUGE ANOMALIES IN HIGHER DIMENSIONS[1]

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## ABSTRACT

Global gauge anomalies in higher dimensions are investigated. It is shown that $Z_3$ global anomalies are absent. However there are non-trivial $Z_2$ global anomalies in $(4k+2)$ dimensions.

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As it was first realized by Witten [1], an SU(2), (or Sp($n$)) theory in four dimensions with a left doublet of fermions has a global gauge anomaly in the sense that the effective action changes sign under gauge transformations $g: S^4 \to$ SU(2), belonging in the non-trivial class of $\pi_4(\mathrm{SU}(2)) \approx Z_2$. The theory is then inconsistent because no global definition of the sign of the fermionic determinant is possible, rendering the theory ill-defined; otherwise, stated the path integral vanishes identically.

The authors of Ref. [2], among others, proposed a beautiful way of calculating the global gauge anomaly, making use of the perturbative one.

Known examples of global gauge anomalies occur only in $4k$ dimensions and are always of the $Z_2$ type, that is the global ambiguity in the definition of the Weyl determinant is a sign. Recently in Ref. [3], global gauge anomalies in higher dimensions were re-examined and an algorithm to construct convenient perturbative anomaly-free representations of SU(n) in $2n$-dimensions coming from the reduction of complex representations of SU(n+1) was given.

In this letter, examples are presented of $Z_3$ global gauge anomalies as well as global anomalies occurring in $4k+2$ dimensions. We start by reviewing briefly the main facts about the relation between global and perturbative gauge anomalies.

Consider a gauge theory with gauge group H in $2n$-dimensions. If $\pi_{2n}$ (H) is non-trivial then the theory has a potential global gauge anomaly[2]. Imagine now embedding H in a group G$\supset H$, such that G has an irreducible perturbative anomaly, (i.e. $\pi_{2n+1}$ (G) $= Z$) and with $\pi_{2n}$ (G) being trivial. Then if we consider a gauge transformation which belongs to a non-trivial class of $\pi_{2n}$ (H), we can connect it continuously to a trivial one escaping out of H into G. Extending the H gauge field trivially to a G gauge field, the change in the effective action of the fermions under this gauge transformation is given by the Wess-Zumino term of the group G.

The Wess-Zumino term can be written as an integral over a $(2n+1)$-disc D with $\partial D = S^{2n}$:

$$\Gamma(g, A, F) = 2\pi \int_D \gamma(g, A, F) \ , \tag{1}$$

where $\gamma$ is a closed but not exact $(2n+1)$-form (an element of $H^{2n+1}$ (G, $Z$)). Consistency in the choice of D requires that:

$$\int_{S^{2n+1}} \gamma(g, A, F) = m, \quad m \in \boldsymbol{Z} \tag{2}$$

Considering now an H gauge transformation $h: S^{2n} \to$ H, we can extend it to a G transformation, g: D $\to$ G such that the restriction of $g$ on $\partial$ D is $h$. The issue of global gauge anomalies is of interest only when the perturbative anomalies cancel. By choosing the H representation free of possible perturbative H anomalies, the Wess-Zumino term effectively defines a mapping in G/H rather than G. Then $\partial$ D is mapped to a point in G/H so that :

$$\int_D \gamma(g, A, F) = \int_{S^{2n+1}} \gamma(g, A, F) \ , \tag{3}$$

[2]For all Lie groups, $\pi_{2n}$ (H) is either trivial or torsion.

and the fermionic measure transforms as:

$$D\mu(g\psi) = D\mu(\psi) \exp\left(2\pi i \int_{S^{2n+1}} \gamma(g, A, F)\right) \tag{4}$$

The last integral in (3) is a homotopy invariant since $d\gamma = 0$, providing a map from $\pi_{2n+1}$ (G/H) $\to Z$.

In the case of the SU(2) anomaly in four dimensions, we choose G = SU(3), H = SU(2). We will try to evaluate (4) using the homotopy theory of fibre bundles. From the exact sequence of the fibration: H $\overset{i}{\longrightarrow}$ G $\overset{p}{\longrightarrow}$ G/H we have,

$$\ldots \to \pi_5[SU(3)] \overset{p^*}{\longrightarrow} \pi_5[SU(3)/SU(2)] \overset{\partial}{\longrightarrow} \pi_4[SU(2)] \overset{i^*}{\longrightarrow} \pi_4[SU(3)] \ldots \tag{5}$$

or

$$0 \to \boldsymbol{Z} \overset{p^*}{\longrightarrow} \boldsymbol{Z} \overset{\partial}{\longrightarrow} \boldsymbol{Z}_2 \; s \overset{i^*}{\longrightarrow} 0 \; . \tag{6}$$

The map $p^*$ is induced by the projection $p$ of the bundle, so that given a homotopy class of maps $f: S^n \to$ G, $p^*$ gives the homotopy class corresponding to the maps $p \circ f$. Similarly $i^*$ is induced by the injection i of the bundle. The sequence (5) dictates that Im$\partial = \boldsymbol{Z}^2$, Ker $\partial = 2\boldsymbol{Z}$ = Im $p^*$. This means that $\partial$ maps odd elements of $\pi_5$ [SU(3)/SU(2)] to the trivial element in $\pi_4$ [SU(2)], and even elements to the non-trivial element in $\pi_4$ [SU(2)]. Since Im$p^* = \boldsymbol{Z}$, $p^*$ maps an element of $\pi^5$ [SU(3)] with winding number $k$ to the corresponding element of $\pi_5$ [SU(3)/SU(2)] with winding number $2k$. In particular $1 \to 2$. So that if $g_1$ is the generator of $\pi_5$ [SU(3)] and $\hat{g}$ the generator of $\pi^5$ [SU(3)/SU(2)], then $g_1 = \hat{g}^2$. This means that by doing the $\hat{g}$ action twice we obtain the action of the element $g_1$. For the generator $g_1$ of $\pi_5$ [SU(3)] we normalize: $\int_{S^5} \gamma(g_1, A, F) = 1$. Let $\hat{h}$ be the generator of $\pi_4$ [SU(2)]. We can extend it to $\hat{g}$ in SU(3) such that $\hat{g} = \hat{h}$ on $\partial D = S^4$, $\hat{g}$ being a map : $S^5 \to$ SU(3)/SU(2). If we choose $\hat{g}$ to correspond to the generator of $\pi_5$ [SU(3)/SU(2)], since from (6) $\hat{g}^2 = g_1$ then :

$$1 = \int \gamma(g_1) = \int \gamma(\hat{g}^2) = 2 \int \gamma(\hat{g}) = (2/2\pi) Q(\hat{g}) \to Q(\hat{g}) = \pi \; . \tag{7}$$

$Q$ is the global anomaly and we can conclude that $D\mu(\hat{h}\psi) = D\mu(\psi) \exp[iQ(\hat{g})] = -D\mu(\psi)$.

The situation can be easily generalized to arbitrary G,H satisfying: $\pi_{2n}$ (H) $\neq 0, \pi_{2n+1}$ (G) $= \boldsymbol{Z}, \pi_{2n}$ (G) $= 0$, $\pi_{2n+1}$ (G/H) $\boldsymbol{Z}$. From the exact sequence:

$$\ldots \to \pi_{2n+1}(G) \overset{p^*}{\longrightarrow} \pi_{2n+1}(G/H) \overset{\partial}{\longrightarrow} \pi_{2n}(H) \overset{i^*}{\longrightarrow} \pi_{2n}(G) \to \ldots \tag{8}$$

we conclude that the global gauge anomaly under a transformation by the generator of $\pi_{2n}(H)$ is given by $\exp(iQ)$,

$$Q = 2\pi \int \gamma(\hat{g}, A, F)_R = 2\pi A_R / N \; , \tag{9}$$

where R is the G-fermion representation, $A_R$ is defined by :

$$Tr[F^{n+1}]_R = A_R \, \mathrm{tr}[F^{n+1}]_f \; + \; \text{lower traces} \; . \tag{10}$$

f is the fundamental representation of G and the lower traces correspond to locally exact forms, not contributing to the anomaly when they are integrated over the sphere. Finally N is defined by ker $\partial$ = Im $p^*$ = N$\boldsymbol{Z}$, the group of integer multiples of N, which means that $\hat{g}^N = g_1$ where $g_1$ is the generator of $\pi_{2n+1}$ [G] while $\hat{g}$ is the generator of $\pi_{2n+1}$ [G/H].

This procedure is straightforwardly extended to the case where $\pi_{2n+1}$ (G/H) contains torsion too[3].

We now proceed to analyze the previously mentioned examples.

The first one deals with a $G_2$ gauge theory in six dimensions. $G_2$ has a reducible perturbative anomaly in six dimensions. As is well known [4], in $4k+2$ dimensions the anomaly cancellation works only among L-R fermion representations. The first step is to construct $G_2$ representations free of perturbative anomalies. For a representation R of $G_2$ we know that, (see [5],[6]),

$$\mathrm{Tr}_R[F^4] = D(R)\ \mathrm{tr}_f[F^2]\ \mathrm{tr}_f[F^2]\ , \tag{11}$$

$$D(R) = \frac{1}{16}\ \ d(R) I_2(R)[3I_2(R) - 8]\ . \tag{12}$$

$d(R)$ is the dimension of $R$, f denotes the fundamental representation, $I_2(R)$ is the second Casimir invariant of $G_2$ and $D(R)$ is normalized such that the fundamental representation (0,1), has $D(0,1) = 7$, $D(R)$ being always an integer. $G_2$ is the maximal subgroup of Spin(7) and Spin(7) can be embedded in SU(7). So, choosing G = SU(7), H = $G_2$ we have the following exact homotopy sequence:

$$\ldots \to \pi_7[SU(7)] \overset{p^*}{\longrightarrow} \pi_7[SU(7)/G_2] \overset{\partial}{\longrightarrow} \pi_6[G_2] \overset{i^*}{\longrightarrow} \pi_6[SU(7)] \to \ldots \tag{13}$$

or

$$0 \to \boldsymbol{Z} \overset{p^*}{\longrightarrow} \pi_7[SU(7)/G_2] \overset{\partial}{\longrightarrow} \boldsymbol{Z}_3 \ \to\ 0\ . \tag{14}$$

Using some known facts about the coset space Spin(7)/$G_2$ = $S^7$ [7], it can be shown that Im $p^* = 3\boldsymbol{Z}$. Consequently, under a gauge transformation generated by the generator g of $\pi_6$ [$G_2$], the fermionic measure transforms as:

$$D\mu(g\psi)_R = D\mu(\psi)_R \exp(2\pi\ iA_{\hat{R}}/3) \tag{15}$$

where $A_{\hat{R}}$ is the leading anomaly coefficient of the SU(7) representation $\hat{R}$ which reduces to the anomaly free $G_2$ representation $R$ through the embedding: $G_2 \subset$ Spin(7) $\subset$ SU(7). We will analyze $G_2$ representations $R^k$, having a Dynkin index $(0,k)$, $k \geq 1$. The representation $F^k \equiv 7R^k_L + D(R^k)R^1_R$ is free from perturbative $G^2$ anomalies for every $k \in \boldsymbol{Z}^+$. If we denote by $[k]$ the $k$-index symmetric tensor representation of SU(7), then:

$$SU(7)\ \ni\ [2]_L \to R^2_L \in G_2\ , \tag{16a}$$

$$SU(7)\ \ni\ [3]_L + [1]_R \to R^3_L \in G_2\ , \tag{16b}$$

[3]It is trivial to show, assuming $\pi_{2n}$ (G) =0, $\pi_{2n+1}$ (G) = $\boldsymbol{Z}$, that $\pi_{2n+1}$ (G/H) not only contains $\boldsymbol{Z}$ but also that $\partial$ is mapping it non-trivially into $\pi_{2n}$ (H).

$$SU(7)\ \ni\ [k]_L + [k-2]_R \rightarrow R_L^k \in G_2 \tag{16c}$$

modulo L-R symmetric representations which do not contribute to the global anomaly. Taking into account the following facts :

$$A([k]) = 1 \text{ (mod 3)} \ \text{ for } \ k = 9n+1, \quad n \in \ \boldsymbol{Z}_0^+ \tag{17}$$

$$D(R^k) = 1 \text{ (mod 3)} \ \text{ for } \ k = 9n+1\ ,\ n \in \ \boldsymbol{Z}_0^+ \tag{18}$$

$$D(R^k) = 2 \text{ (mod 3)} \ \text{ for } \ k = 9n+3\ ,\ n \in \ \boldsymbol{Z}_0^+ \tag{19}$$

and both zero otherwise, we can conclude that all representations free of the perturbative anomaly are also free of the global anomaly. There is also an independent argument towards this fact. Since the product of $G_2$ representations $(1,0)\otimes(0,1)$ includes the identity the global anomaly can be at most $Z_2$. Since $Z_2$ is not a subgroup of $Z_3$ the anomaly vanishes[4].

Let us now consider an SU(2) gauge theory and choose G = SU(4).

The exact homotopy sequence in this case is:

$$0 \rightarrow \boldsymbol{Z} \overset{p^*}{\rightarrow} \pi_7[SU(4)/SU(2)] \overset{\partial}{\rightarrow} \boldsymbol{Z}_{12} \ \rightarrow \ 0 \tag{20}$$

$\pi_7$ [SU(4)/SU(2)] = $\pi_7$ [U(4)/U(2)], and U(4)/U(2) is isomorphic to $S^5\times S^7$ so $\pi_7$ [SU(4)/SU(2)] = $\boldsymbol{Z}+\boldsymbol{Z}_2$ and it can be inferred that Im $p^* = 12\boldsymbol{Z}$. Consequently, the phase change in the fermionic measure is $\exp(2\pi i A_R/12)$ in this case. The perturbative anomaly in six dimensions of an SU(2) representation $[k]$ is related to the anomaly of the fundamental f by, [5]

$$\mathrm{Tr}_k[X^4] = D_2(k)\ \mathrm{tr}_f[X^2]\mathrm{tr}_f[X^2]\ , \tag{21}$$

$$D_2(k) = \frac{1}{30}\ k(k+1)(k+2)(3k^2+6k-4)\ , \tag{22}$$

where $[k]$ denotes the $k$-index symmetric tensor representation, and the leading SU(4) anomaly is given by [5]

$$A(k) = \frac{1}{840}k(k+1)(k+2)(k+3)(k+4)(k^2+4k+2)\ . \tag{23}$$

Since $[k]_L + D_2(k)[1]_R$ is a representation free of perturbative SU(2) anomalies, we can embed it in SU(4):

$$SU(4) \ni [k]_L + [k-1]_R + [k-2]_L + D_2(k)[1]_R \rightarrow [k]_L + D_2(k)[1]_R \in SU(2) \tag{24}$$

modulo L-R symmetric representations. We can easily now verify that for $k = 8n+2, 8n+4$, $n \ \in \ \boldsymbol{Z}^+$ only, we end up with a phase which is $e^{i\pi} = -1$ and it is zero for any other $k$. The anomaly here is a $\boldsymbol{Z}_2$ anomaly.

A similar analysis in a SU(3) gauge theory in six dimensions, which has a potential $\boldsymbol{Z}_6$ global anomaly, reveals that any SU(3) representation free of perturbative anomalies is also free of global anomalies.

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[4]I would like to thank A. Polychronakos for providing this independent argument.

Looking now in eight dimensions we can easily find, using the procedure above, that an SU(2) theory has a $\boldsymbol{Z}_2$ anomaly only for the $[8n+1]$ representations (the theory is perturbative anomaly free), and an SU(3) theory has no global anomaly at all.

For an SU(4) theory in eight dimensions, we can deduce that $[k]_L + D(k)\ [1]_R$ with [5]

$$D(k) = \frac{1}{2520} k(k+1)(k+2)^2(k+3)(k+4)(3k^2+12k-8) \tag{25}$$

which is free of perturbative anomalies, has a $\boldsymbol{Z}_2$ global anomaly for $k = 8n+2, \quad n \in \boldsymbol{Z}_0^+$, and it is anomaly free otherwise. This fact was realised for the first few of them in [2].

For a $G_2$ theory in eight dimensions, (which is perturbative anomaly free), and considering representations of the form $(0,k)$, we can deduce that only representations $(0, 8n+1)$, $(0, 8n+2)$ have a global $\boldsymbol{Z}_2$ anomaly, the others being anomaly free.

Finally an SU(2) theory in ten dimensions, despite the fact that it can have a potential $\boldsymbol{Z}_{15}$ global anomaly, is anomaly free for any representation free of perturbative anomalies.

We can conclude that non-trivial global anomalies also exist in $(4k+2)$ dimensions.

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