Chapter 1

TRANSPORT IN ONE DIMENSIONAL QUANTUM SYSTEMS

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Abstract   In this Chapter, we present recent theoretical developments on the finite temperature transport of one dimensional electronic and magnetic quantum systems as described by a variety of prototype models. In particular, we discuss the unconventional transport and dynamic - spin, electrical, thermal - properties implied by the integrability of models as the spin-1/2 Heisenberg chain or Hubbard. Furthermore, we address the implication of these developments to experimental studies and theoretical descriptions by low energy effective theories.

Keywords: one dimensional quantum many body systems,
              transport, integrability

1.   Introduction

The electronic and magnetic properties of reduced dimensionality materials are significantly modified by strong correlation effects. In particular, over the last few years, the physics of quasi-one dimensional electronic systems, has been the focus of an ever increasing number of theoretical and experimental studies. They are realized as three dimen-

sional (3D) compounds composed of weakly interacting chains or, in a latest and very promising development, as monoatomic width chains fabricated by self-assembly on surfaces.

Experimentally, recent studies made possible by the synthesis of new families of compounds characterized by very weak interchain coupling and low disorder, indicate unconventional transport and dynamic behavior; for example, unusually high thermal conductivity in quasi-one dimensional magnetic compounds [1, 2, 3], ballistic spin transport in magnetic chains [4, 5] or optical conductivity in quasi-1D organic conductors, showing a low frequency very narrow "Drude peak" even at relatively high temperatures [6, 7].

Theoretically, it is well known that one dimensional (1D) systems of interacting electrons do not follow the phenomenological description of the ordinary Landau Fermi liquids, but rather they are characterized by a novel class of collective quantum states coined Luttinger liquids [8]. Furthermore, it has quite recently been realized that several prototype models commonly used to describe 1D materials imply ideal transport properties (dissipationless) even at high temperatures. This phenomenon is the quantum analogue of transport by nondecaying pulses (solitons) in 1D classical nonlinear integrable systems [9].

Of course, 1D electronic and magnetic systems have, since the sixties, been a favorite playground where theoretical ideas were confronted with experimental results on an ever improving quality and variety of quasi-1D compounds. We are now in a position to claim reliable theoretical analysis on the thermodynamics, quantum phase transitions and spectral functions of prototype many body Hamiltonians used to describe 1D materials. The tools at hand range from exact analytical solutions (e.g. using the Bethe ansatz (BA) method) [10], the low energy Luttinger liquid approach and powerful numerical simulation techniques as the Quantum Monte Carlo (QMC) [11] and Density Matrix Renormalization Group (DMRG) [12] method. In particular, the ground-state properties as well as the low-temperature behavior of the correlation functions and of most static quantities in the scaling (universal) regime of Luttinger liquids have been extensively studied in recent decades and are well understood by now [13, 14, 15, 16].

On the other hand, although most experimentally relevant, less studied and understood are the transport and dynamic properties of 1D interacting electronic or magnetic systems. Paradoxically, while the equilibrium properties of prototype integrable models as the spin-1/2 Heisenberg or Hubbard model have been extensively analyzed, their finite temperature transport has attracted little attention; most studies

till a few years ago, have basically focused on the low energy description in terms of the Luttinger model. A noticeable exception has been the issue of diffusive versus ballistic behavior and thermal conductivity of spin chains, a long standing and controversial issue. The difficulties encountered with the transport quantities can be attributed to the fact that the scattering and dissipation in clean 1D fermionic systems are not dominated by low-energy processes and thus the transport properties are not universal.

Presently, transport properties are at the focus of intense theoretical activity; in particular, prototype integrable models (as the Heisenberg, t-J, Hubbard, nonlinear-σ) are studied by exact analytical techniques (e.g. Bethe ansatz, form factor method) and numerical simulations. However, the complexity of these methods often renders the resulting behavior still controversial. Furthermore, the transport of quasi-one dimensional systems is (re-) analyzed within the effective Luttinger liquid theory or by semiclassical, Boltzmann type, approaches.

It is fair to say that the study of finite temperature/frequency conductivities in strongly correlated systems presents at the moment fundamental conceptual as well as technical challenges. Development of new analytical and numerical simulation techniques is required, as well as progress on the basic understanding of scattering mechanisms and their effects.

In the following, we will mostly concentrate on the conductivity of bulk, clean systems where the scattering mechanism is due to electronic or magnetic interactions (Umklapp processes). In particular, we will not address issues on the transport of mesoscopic systems (e.g. nanowires, nanotubes) or other dissipation mechanisms as coupling to phonons or disorder.

In section 2 we start by presenting some elements of linear response theory (or Kubo formalism), the theoretical framework commonly used for describing transport properties. Then, in section 3 and 4 we continue with a presentation of the state of the transport properties of prototype systems, in particular the Heisenberg and Hubbard model. In section 5, we present a short overview of alternative approaches based on low energy effective field theories as the Luttinger liquid, sine-Gordon and quantum nonlinear-σ models. Finally, in section 6, we close with a critical assessment of the present status and a discussion of open issues.

This presentation is definitely not an exhaustive account of theoretical studies on the transport properties of one dimensional quantum systems but it rather aims at presenting a coherent and self-contained view of some recent developments.

2.   Linear response theory

In this section we introduce the basic definitions and concepts that will be used in the later development. The framework of most transport studies is linear response theory where the conductivities are given in terms of finite temperature (T) dynamic correlations calculated at thermodynamic equilibrium [17]. For instance, the real part of the electrical conductivity at frequency ω is given by the corresponding dynamic current correlation $\chi_{jj}(\omega)$,

$$\sigma'(\omega) = 2\pi D\delta(\omega) + \sigma_{reg}(\omega) \tag{1.1}$$

$$\sigma_{reg}(\omega) = \Re\frac{1}{i\omega}\chi_{jj}(\omega) \tag{1.2}$$

$$\chi_{jj}(\omega) = i\int_0^{+\infty} dt e^{izt}\langle[j(t),j]\rangle, \quad z = \omega + i\eta \tag{1.3}$$

with j the appropriate current operator. In a spectral representation the conductivity is,

$$\sigma_{reg}(\omega) = \frac{1 - e^{-\beta\omega}}{\omega}\frac{\pi}{L}\sum_n p_n \sum_{m\neq n} |\langle n \mid j \mid m \rangle|^2 \, \delta(\omega - (\epsilon_m - \epsilon_n)), \tag{1.4}$$

$\mid n\rangle, \epsilon_n$ denoting the eigenstates and eigenvalues of the Hamiltonian, p_n the corresponding Boltzmann weights and β the inverse temperature (in the following we take $\hbar = \kappa_B = e = 1$); the dc conductivity is given by the limit $\sigma_{dc} = \sigma_{reg}(\omega \to 0)$. We will mostly discuss one dimensional tight-binding models on L sites where the current operator does not commute with the Hamiltonian.

To define the current operators we use the continuity equations of charge, magnetization or energy for the electrical, magnetic and thermal conductivity, respectively. We will explicitly present them below in the discussion of the Heisenberg and Hubbard models.

A quantity that presently attracts particular attention is the prefactor D of the $\delta-$function, named the Drude weight or charge stiffness. This quantity was introduced by W. Kohn in 1964 as a criterion of (ideal) conducting or insulating behavior [18] at $T = 0$ in the context of the Mott-Hubbard transition. This meaning becomes clear by noting that D is also the prefactor of the low frequency, imaginary (reactive - nondissipative) part of the conductivity,

$$D = \frac{1}{2}[\omega\sigma''(\omega)]_{\omega\to 0} = \frac{1}{L}\left(\frac{1}{2}\langle -T\rangle - \sum_n p_n \sum_{m\neq n}\frac{|\langle n \mid j \mid m \rangle|^2}{\epsilon_m - \epsilon_n}\right); \tag{1.5}$$

here $\langle T \rangle$ denotes the thermal expectation value of the kinetic energy, generalizing the zero temperature expression to $T > 0$ by considering a thermal average. Thus, a finite Drude weight implies an "ideal conductor", a freely accelerating system. The second definition of the Drude weight follows from the familiar optical sum-rule [19, 20, 21] using eq.(1.1),

$$\int_{-\infty}^{+\infty} \sigma'(\omega)d\omega = \frac{\pi}{L}\langle -T \rangle, \tag{1.6}$$

with the average value of the kinetic energy replacing for nearest neighbor hopping tight binding models the usual ratio of density over mass of the carriers for systems in the continuum.

At $T = 0$ the Drude weight $D_0 = D(T = 0)$ is the central quantity determining charge transport. As already formulated by Kohn in a very physical way, D_0 can also be expressed directly as the sensitivity of the ground state energy ϵ_0 to an applied flux $\phi = eA$ ($e = 1$),

$$D_0 = \frac{1}{2L}\frac{\partial^2 \epsilon_0}{\partial \phi^2}\big|_{\phi \to 0}. \tag{1.7}$$

For a clean system, since at $T = 0$ there cannot be any dissipation, one expects that $\sigma_{reg}(\omega \to 0) = 0$ and we have to deal with two fundamentally different possibilities with respect to D_0:

$D_0 > 0$ is characteristic of a *conductor* or *metal*,

$D_0 = 0$ characterizes an *insulator*.

The insulating state can originate from a filled electron band (usual band insulator) or for a non-filled band from electron correlations, that is the Mott-Hubbard mechanism; the latter situation is of interest here. Note, that the same criterion of the sensitivity to flux has been applied to disordered systems, in the context of electron localization theory [22].

The theory of the metal-insulator transition solely due to Coulomb repulsion (Mott transition) has been intensively investigated in the last decades by analytical and numerical studies [23] of particular models of correlated electrons and it is one of the better understood parts of the physics of strongly correlated electrons.

At finite temperatures, within the usual Boltzmann theory for weak electron scattering, the relaxation time approximation represents well the low frequency behavior,

$$\sigma(\omega) = \sigma_{dc}/(1 + i\omega\tau), \tag{1.8}$$

where the relaxation time τ depends on the particular scattering mechanism and is in general temperature dependent. In the following, we consider only homogeneous systems without any disorder, so the relevant processes in the solid state are electron-phonon scattering and the electron-electron (Coulomb) repulsion. When the latter becomes strong it is expected to dominate also the transport quantities.

Even in a metal with $D_0 > 0$ it is not evident which is the relevant scattering process determining $\tau(T)$ and $\sigma_{dc}(T)$. In the absence of disorder and neglecting the electron-phonon coupling the standard theory of purely electron-electron scattering would state that one needs Umklapp scattering processes to obtain a finite τ. That is, the relevant electron Hamiltonian includes the kinetic energy H_{kin}, the lattice periodic potential V and the electron-electron interaction H_{int},

$$H = H_{kin} + V + H_{int}. \tag{1.9}$$

Then, in general, the electronic current density j is not conserved in an Umklapp scattering process as the sum of ingoing electron momenta equals the sum of outgoing ones only up to a nonzero reciprocal vector G, $\sum_i k_i = mG$. In other words, the noncommutativity of the current with the Hamiltonian, $[H, j] \neq 0$, leads to current relaxation and thus, by the fluctuation-dissipation theorem, to dissipation. The interplay of V and H_{int}, however, turns out to be fairly involved in the case of strong electron-electron repulsion. This will become clear in examples of integrable tight binding models of interacting systems that we will discuss below, which have anomalous (diverging) transport coefficients.

Experiments on many novel materials, - strange metals - with correlated electrons, question the validity of the concept of a current relaxation rate $1/\tau$. Prominent examples are the superconducting cuprates with very anisotropic, nearly planar, transport [23] where the experimentally observed $\sigma(\omega)$ in the normal state can be phenomenologically described only by strongly frequency (and temperature T) dependent $\tau(\omega, T)$. The experiments on $\sigma(\omega)$ in quasi-1D systems are covered elsewhere [7].

With this background, we will now discuss different possible scenarios for the behavior of the $T > 0$ conductivity. A clean metallic system at $T = 0$ is characterized by a $\delta-$function Drude peak and a finite frequency part that vanishes, typically with a power law dependence, implying zero dc regular conductivity. In the common sense scenario, at finite temperatures the $\delta-$function broadens to a "Drude peak" of width inversely proportional to a characteristic scattering time and thus a finite $\omega \to 0$ limit implying a finite dc conductivity. The scattering mechanisms can
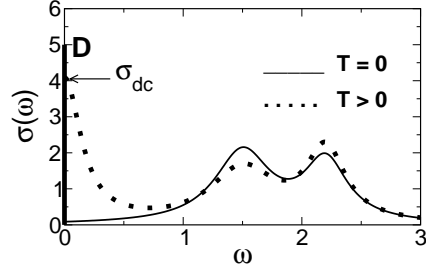
Figure 1.1. Schematic representation of the typical behavior of the conductivity for a clean metal.
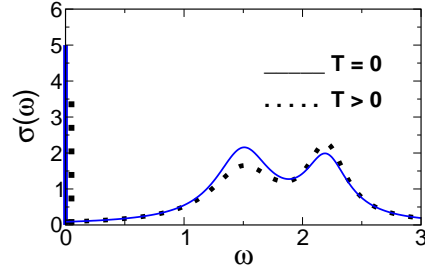

Figure 1.2. Illustration of the conductivity of a clean metal remaining an ideal conductor at finite temperatures.

be intrinsic, due to interactions, or extrinsic due to coupling to other excitations, phonons, magnons etc. This typical behavior is shown in Fig. 1.1.

Actually, for a finite size system (as often studied in numerical simulations) D is nonzero even at finite T; it only goes to zero, typically exponentially fast, as the system size tends to infinity. Physically, this expresses the situation where the thermal scattering length is less than the system size.

But it is also possible that constraints on the scattering mechanisms limit the current decay, so that the system remains an ideal conductor ($D > 0$) even at $T > 0$. A schematic representation of a system remaining an "ideal conductor" at finite T is shown in Fig. 1.2.

In a system with disorder, D vanishes even at zero temperature and the dc residual conductivity is finite (provided the disorder is not strong enough to produce localization).

For an insulating system, e.g. due to interactions or the band structure as we discussed above, D vanishes at zero temperature; in the conventional case, D remains zero at $T > 0$, while activated carriers scattered via different processes give rise to a finite dc conductivity. But it is also possible that D becomes finite, turning a $T = 0$ insulator to an ideal conductor; for instance, a system of independent particles (e.g. one described within a mean field theory scheme), insulating due to the band structure, turns to an ideal conductor at $T > 0$. Finally, it is also conceivable that both D and σ_{dc} remain zero at $T > 0$, a system that can be called an "ideal insulator".

To the above scenaria we should add the possibility that the low frequency conductivity at finite temperatures is anomalous, e.g. diverging

as a power law of the frequency, resulting in an infinite dc conductivity. Actually, as we will discuss later (Discussion section), this kind of behavior is fairly common in classical one dimensional nonlinear systems.

Thus, the first step in characterizing a system is the evaluation of the Drude weight at $T = 0$ in order to find out whether the system is conducting or insulating. The peculiarity that has recently been noticed is that most prototype models, assumed faithful representations of the physics of several quasi-one dimensional materials, have finite Drude weight also at finite temperatures (even $T \to \infty$), thus implying intrinsically ideal conductivity. In other words, interactions do not present a sufficient scattering mechanism to turn these systems into normal conductors. This behavior is unlike the one observed in the higher dimensional version of the same models, that become normal conductors at finite temperatures [24]. This unconventional behavior has been attributed to the integrability of these models.

To evaluate the Drude weight is not an easy matter as, although frequency independent, it represents a transport property and thus it cannot be obtained via a thermodynamic derivative (e.g. of the free energy). Direct calculation using the optical sum rule eq.(1.6) is obviously involved requiring the value of all current matrix elements. A very convenient and physical formulation is the one by W. Kohn, eq.(1.7), that generalized at finite temperatures [25] reads,

$$D = \frac{1}{2L} \sum_n p_n \frac{\partial^2 \epsilon_n(\phi)}{\partial \phi^2}|_{\phi \to 0}. \tag{1.10}$$

By considering the change of the free energy as a function of flux (that vanishes in the thermodynamic limit as it is proportional to the susceptibility for persistent currents) we can also arrive at an expression for the Drude weight as the long time asymptotic value of current-current correlations [26],

$$D = \frac{\beta}{2L} \sum_n p_n \langle n|j|n \rangle^2 = \frac{\beta}{2L} \langle j(t)j \rangle_{t \to \infty} \equiv \beta C_{jj}. \tag{1.11}$$

As an example, for a 1D tight binding free spinless fermion system with nearest neighbor hopping t, the application of a flux ϕ modifies the single particle dispersion to $\epsilon_k = -2t \cos(k + \phi)$ giving,

$$D_0 = \frac{t}{\pi} \sin(\pi n) = N(\epsilon_F)j_F^2$$

$$D(T) \sim D_0 - \frac{\pi t}{12}(\frac{T}{t})^2 \quad (n = \frac{1}{2}). \tag{1.12}$$

Here, n is the fermion density, $N(\epsilon_F)$ the density of states and j_F the current at the Fermi energy. Notice the quadratic decrease with temperature of the Drude weight that, as we will see later, it is generic even for interacting one dimensional fermionic systems out of half-filling.

In the recent literature, that we will discuss below, the Drude weight of integrable systems is evaluated by the BA technique at zero or finite temperatures using the Kohn expression (1.10). The difficulty in this approach is the need for the estimation of finite size energy corrections of the order of $1/L$, a rather subtle procedure within this method.

Another approach, proved particularly efficient in establishing that systems with a finite Drude weight at finite temperature exist, uses an inequality proposed by Mazur [27]. This inequality states that if a system is characterized by conservation laws Q_n then:

$$\lim_{\mathcal{T} \to \infty} \frac{1}{\mathcal{T}} \int_0^{\mathcal{T}} \langle A(t)A \rangle dt \geq \sum_n \frac{\langle AQ_n \rangle^2}{\langle Q_n^2 \rangle}. \tag{1.13}$$

Here $\langle \ \ \rangle$ denotes a thermodynamic average, the sum is over a subset of conserved quantities Q_n orthogonal to each other in the sense $\langle Q_n Q_m \rangle = \langle Q_n^2 \rangle \delta_{n,m}$, $A^\dagger = A$ and we take $\langle A \rangle = 0$.

Thus, for time correlations $\langle A(t)A \rangle$ with non-singular low frequency behavior we can obtain a bound for $C_{AA} = \lim_{t \to \infty} \langle A(t)A \rangle$,

$$C_{AA} \geq \sum_n \frac{\langle AQ_n \rangle^2}{\langle Q_n^2 \rangle}. \tag{1.14}$$

For integrable systems, such as the spin-1/2 Heisenberg or Hubbard model that are known to possess nontrivial conservation laws because of their integrability, useful bounds can be obtained by considering just the first non-trivial conservation law. We should stress however that this approach has not provided yet a complete picture of the Drude weight behavior as we will discuss below in concrete examples.

Finally, another argument relating the behavior of the Drude weight to the (non-) integrability of a model is by the use of Random Matrix Theory [28, 29, 25]. It is known that integrable systems are characterized by energy level crossings upon varying a parameter and so Poisson statistics in the energy level spacing; thus it can be argued that the typical value of diagonal current matrix elements (slope of energy levels with respect to an infinitesimal flux) is of the order of one, plausibly implying a finite Drude weight according to eq.(1.11). On the contrary, nonintegrable systems, due to level repulsion, are described by Wigner or GOE statistics and thus the characteristic value of diagonal current

matrix elements is of the order of e^{-L} (inversely proportional to the density of many body states) implying now a vanishing Drude weight as $L \to \infty$.

Besides electrical transport, the thermal conductivity of 1D systems has recently attracted particular interest; within linear response theory it is given by the analogous Green-Kubo formula expressed in terms of the energy current - energy current dynamic correlation function,

$$\kappa(\omega) = \Re \frac{\beta}{i\omega} \chi_{j^E j^E}(\omega). \tag{1.15}$$

Unlike the conductivity, there is no "mechanical force" (as the flux ϕ) that can be applied to the system in order to deduce expressions similar to the Drude weight, but the long time asymptotic value of energy current correlations has an analogous meaning.

Finally, in magnetic systems, the "spin conductivity" (spin diffusion constant) can be probed, for instance, by NMR experiments that measure at high temperatures the Fourier transform of spin-spin autocorrelations at the Larmor frequency ω_N,

$$S(\omega_N) = \int_{-\infty}^{+\infty} dt \int dq e^{i\omega_N t} \langle S_q^z(t) S_{-q}^z \rangle. \tag{1.16}$$

By using the continuity equation,

$$\omega^2 \langle S_q^z S_{-q}^z \rangle_\omega = q^2 \langle j_q^z j_{-q}^z \rangle_\omega \tag{1.17}$$

for a system where the total spin $z-$component is conserved, the spin-spin dynamic correlations can be analyzed via the corresponding spin-current correlations in analogy to electrical transport [30]; the role of local charge is played by the $z-$ component of the local magnetization (see next section for a more detailed discussion on this point).

We will now briefly discuss different methods, analytical and numerical, that are available for the study of finite temperature dynamic correlations in strongly interacting systems. Among the analytical approaches that have been used for the study of transport and dynamic properties of 1D systems, each has its own advantages and drawbacks. The traditional memory function approach [31] provides a complete picture of the temperature/frequency dependence but it is a perturbative method based on the assumption of a regular relaxation behavior that might be dangerous in 1D systems. The high temperature moment expansion provides useful information on the possibility of anomalous transport

but the extraction of transport coefficients is also based on the phenomenological assumption of regular, diffusive behavior [32]. Progress in the exact evaluation of dynamic correlations in integrable systems has recently been achieved in the calculation of the Drude weight by the Bethe ansatz technique and of the frequency dependent conductivity by the form factor method. The Drude weight studies however are still controversial as they involve the calculation of finite size corrections, while the form factor approach has so far been limited to the calculation of zero temperature correlations and mostly in gapped systems. It is expected however that progress in BA techniques will provide a full picture of the dynamic properties of integrable systems. It is amusing to remark the paradoxical situation where the only strongly correlated systems for which we can probably have a complete solution of their dynamics are the integrable ones, which however, exactly because of their integrability, show unconventional behavior.

Among numerical simulation techniques, the ED (exact diagonalization) provides exact answers over the full temperature/frequency range but of course only on finite size systems [33]. Due to the exponentially growing size of Hilbert space, this limits the size of systems that can be studied to only about 20 to 30 sites, depending on the complexity of the Hamiltonian. We should also remark that, in principle, the full excitation spectrum is required for the evaluation of finite temperature correlations [1] Furthermore, the obtained frequency spectra are discrete, δ−functions corresponding to transitions between energy levels, so that some ad-hoc smoothing procedure is needed; this is particularly crucial in attempting to extract the low frequency behavior. Nevertheless, finite size scaling in 1D systems can provide very useful hints on the macroscopic behavior, particularly at high temperatures where all energy levels are involved. This regime is the most favorable in attempting to simulate the physical situation where the scattering length is less or comparable to the system size.

The Quantum Monte Carlo techniques allow the study of far larger systems and they provide directly the dynamic correlations at finite temperatures but in imaginary time [35]. By analytical continuation, using for instance the Maximum Entropy procedure, one is able in principle to extract the main features of the frequency dependence; experience shows however, that fine issues as the temperature dependence of the Drude

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[1]In a recent advance, finite temperature dynamic correlations for a prototype model have been successfully evaluated using only one quantum state (microcanonical ensemble) [34].

weight or the presence of diffusive behavior which is a low frequency property, are difficult to establish reliably.

Finally, the DMRG method that has been so successful in the study of ground state and thermodynamic properties of 1D systems, has only recently been extended to the reliable study of zero temperature conductivities in gapped systems [36, 37]. At finite temperatures it is also possible to obtain very high accuracy data on autocorrelation functions in imaginary time by the use of the transfer matrix DMRG [38]. However, similarly to QMC methods, it is very difficult to extract subtle information on the finite T dynamics because of the extremely singular nature of analytic continuation that hides the useful information even for practically exact imaginary time data.

3.   Heisenberg model

The prototype model for the description of localized magnetism is the Heisenberg model. For a one dimensional system the minimal Hamiltonian describing magnetic insulators is,

$$H = \sum_l h_l = J \sum_{l=1}^{L} (S_l^x S_{l+1}^x + S_l^y S_{l+1}^y + \Delta S_l^z S_{l+1}^z) \qquad (1.18)$$

where S_l^α $(\alpha = x, y, z)$ are spin operators on site l ranging from the most quantum case of spin S=1/2 to classical unit vectors. For S=1/2 the system is integrable by the Bethe ansatz method and its ground state, thermodynamic properties and elementary excitations have well been established [10]. As a brief reminder to the discussion that follows, note that for $J > 0$, $\Delta > 0$ corresponds to an antiferromagnetic coupling while $\Delta < 0$ to a ferromagnetic one; a canonical transformation maps $H(\Delta)$ to $-H(-\Delta)$. Further, the anisotropy parameter Δ describes two regimes, the "easy-plane" for $|\Delta| < 1$ or the "easy-axis" for $|\Delta| > 1$. The isotropic case, occuring in most materials for symmetry reasons, corresponds to $\Delta = 1$. For $|\Delta| \leq 1$ the system is gapless and characterized by a linear spectrum at low energies, while for $\Delta > 1$ a gap opens; in particular, at $\Delta = 1$ the elementary excitation spectrum is described by the "des Cloiseaux-Pearson" dispersion $\epsilon_q = \frac{J\pi}{2}|\sin q|$. For $\Delta < -1$ there is a transition to a ferromagnetic ground state.

In general, other types of terms appear in the description of quasi-1D materials such as longer range or on site anisotropy interactions, but in this review we will focus on the prototype model eq.(1.18).

At this point we should mention that the spin-1/2 Heisenberg model is equivalent to a model of interacting spinless fermions (the "t-V" model) obtained by a Jordan-Wigner transformation [39];

$$H = (-t) \sum_{l=1}^{L} (c_l^\dagger c_{l+1} + h.c.) + V \sum_{l=1}^{L} (n_l - \frac{1}{2})(n_{l+1} - \frac{1}{2}), \qquad (1.19)$$

where $c_l(c_l^\dagger)$ denote annihilation (creation) operators of spinless fermions at site l and $n_l = c_l^\dagger c_l$.

The correspondence of parameters is $V/t = 2\Delta$ and the opening of a gap at $\Delta \geq 1$ corresponds to an interaction driven metal-insulator (Mott-Hubbard type) transition.

3.1   Currents and dynamic correlations

Regarding the transport and dynamic properties of the Heisenberg model, three cases have mostly been discussed: the classical one, the spin S=1 and the spin S=1/2; the S=1 case has been extensively analyzed by mapping its low energy physics to a field theory [40], the nonlinear-σ model (see section 5). In connection to experiment, the main issue is the diffusive vs. ballistic character of spin transport as probed for instance by NMR experiments and recently the contribution of magnetic excitations to the thermal conductivity of quasi-one dimensional materials [1].

To discuss magnetic transport, we must first define the relevant spin j^z and energy j^E currents by the continuity equations of the corresponding local spin density S_l^z (provided the total S^z component is conserved) and local energy h_l;

$$S^z = \sum_l S_l^z, \qquad \frac{\partial S_l^z}{\partial t} + \nabla j_l^z = 0, \qquad (1.20)$$

gives for the spin current,

$$j^z = \sum_l j_l^z = J \sum_l (S_l^x S_{l+1}^y - S_l^y S_{l+1}^x). \qquad (1.21)$$

Here and thereafter, $\nabla a_l = a_l - a_{l-1}$ denotes the discrete gradient of a local operator a_l. In general ($\Delta \neq 0$) the spin current does not commute with the Hamiltonian, $[j^z, H] \neq 0$, so that nontrivial relaxation is expected and thus finite spin conductivity at $T > 0$.

Similarly, the energy current j^E is obtained by,

$$j^E = \sum_l j_l^E, \qquad \frac{\partial h_l}{\partial t} + \nabla j_l^E = 0, \qquad (1.22)$$

$$j^E = J\sum_l (S_{l-1}^x S_l^z S_{l+1}^y - S_{l-1}^y S_l^z S_{l+1}^x) + \Delta(S_{l-1}^y S_l^x S_{l+1}^z - S_{l-1}^z S_l^x S_{l+1}^y)$$
$$+ \Delta(S_{l-1}^z S_l^y S_{l+1}^x - S_{l-1}^x S_l^y S_{l+1}^z) \tag{1.23}$$

We will now briefly comment on the framework for discussing spin dynamics and in particular how it is probed by NMR experiments. According to the spin diffusion phenomenology (for a detailed description see ref. [41]) when we consider the (q,ω) correlations of a conserved quantity $A = \sum_l A_l$, such as the magnetization or the energy, it is assumed that it will show a diffusive behavior in the long-time $t \to \infty$, short wavelength $q \to 0$ regime [2]. In the language of dynamic correlation function, diffusive behavior means that the time correlations decay as,

$$\langle\{A_l(t), A_0(0)\}\rangle = 2\chi_A T \int \frac{dq}{2\pi} e^{iql - D_A q^2 |t|} \tag{1.24}$$

where D_A, χ_A are the corresponding diffusion constant and static susceptibility, respectively. For a 1D system, this behavior translates to a characteristic $1/\sqrt{t}$ dependence of the autocorrelation function.

Fourier transforming the above expression we obtain,

$$S_{AA}(q,\omega) = \int_{-\infty}^{+\infty} dt\ e^{i\omega t} \frac{1}{2}\langle\{A_q(t), A_{-q}(0)\}\rangle \sim \frac{\chi_A D_A q^2}{(D_A q^2)^2 + \omega^2}. \tag{1.25}$$

By using the continuity equation (1.20), this Lorentzian form can be further modified to obtain the current-current correlation function,

$$S_{j^A j^A}(q,\omega) \sim \frac{\chi_A D_A \omega^2}{(D_A q^2)^2 + \omega^2} \tag{1.26}$$

which gives the diffusion constant D_A by taking the $q \to 0$ limit first and then $\omega \to 0$.

On the other hand, a ballistic behavior is signaled by a $\delta-$function form, $S_{j^A j^A}(q,\omega) \sim \delta(\omega - cq)$, where c is a characteristic velocity of the excitations. This $\delta-$function peak moves to zero frequency as $q \to 0$ and its weight is proportional to the long time asymptotic of the current-current correlations

$$C_{j^A j^A} = S_{j^A j^A}(q = 0, t \to \infty). \tag{1.27}$$

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[2]This phenomenological statement goes under the name of Ohm's law in the context of electrical transport, Fourier's law for heat or Fick's law for diffusion.

The above anticommutator correlations are related to the imaginary part of the susceptibility $\chi(q,\omega)$, that describes the dissipation, by,

$$S_{AA}(q,\omega) = \coth(\frac{\beta\omega}{2})\chi''_{AA}(q,\omega). \qquad (1.28)$$

In relation to the experimental study of spin dynamics, the NMR has developed to a very powerful tool; for instance, the $1/T_1$ relaxation time is directly related to the spin-spin autocorrelation by,

$$\frac{1}{T_1} \sim |A|^2 \int_{-\infty}^{+\infty} dt \cos(\omega_N t)\langle\{S_l^z(t), S_l^z(0)\}\rangle \qquad (1.29)$$

where $|A|^2$ is the hyperfine coupling [4] and ω_N the Larmor frequency. Using the relation (1.28), $1/T_1$ gives information (in the high temperature limit, $\beta\omega_N \to 0$) on $\chi''(q,\omega)$ as,

$$\frac{1}{T_1} \sim T|A|^2 \sum_q \frac{\chi''(q,\omega_N)}{\omega_N}. \qquad (1.30)$$

The diffusive behavior, characterized by the $1/\sqrt{t}$ decay of the spin correlations, is extracted in an NMR experiment by analyzing the $q \to 0$ contribution [5]. It gives a $1/\sqrt{\omega}_N$ behavior that is detected as a $1/\sqrt{H}$ magnetic field dependence,

$$\frac{1}{T_1} \sim \frac{1}{\sqrt{\omega}_N} \sim \frac{T\chi(q=0)}{\sqrt{D_s H}}, \qquad (1.31)$$

considering that the Larmor frequency $\omega_N \sim H$, D_s being the spin diffusion constant and $\chi(q=0)$ the static susceptibility.

3.2   Spin and energy dynamics

Returning now to the state of spin and energy dynamics, the classical Heisenberg model has been extensively studied by numerical simulations, the first studies dating from the 70's [42]. Nevertheless, the issue of diffusive behavior (even at $T = \infty$ where most simulations are carried out) still seems not totally clear, the energy and spin showing distinctly different dynamics. On the one hand, simulations clearly indicate that energy transport is diffusive [43] but on the other hand, the decay of spin autocorrelations is probably inconsistent with the expected $1/\sqrt{t}$ law [44, 43] exhibiting long-time tails.

On the other extreme, for the fully quantum spin S=1/2 model, the simplest case is the $\Delta = 0$, so called XY limit. Here, the spin current

commutes with the Hamiltonian resulting in ballistic transport; this can also be seen in the fermionic, t-V, version of model that corresponds to free spinless fermions ($V/t = 0$ in eq.(1.19)) where now the charge current is conserved. In the infinite temperature limit ($\beta = 0$) the spin and energy autocorrelations can be calculated analytically using the Jordan-Wigner transformation and are of the form [45]:

$$\langle S_l^z(t) S_l^z \rangle = \frac{1}{4} J_0^2(Jt) \tag{1.32}$$

$$\langle h_l(t) h_l \rangle = \frac{J^2}{8} \left(J_0^2(Jt) + J_1^2(Jt) \right) \tag{1.33}$$

which both behave as $1/t$ for $t \to \infty$, unlike the $1/\sqrt{t}$ form in the diffusion phenomenology (J_0, J_1 are Bessel functions). Actually the $\beta = 0$ limit, often theoretically analyzed for simplicity, is not unrealistic as the magnetic exchange energy J can be of the order of a few Kelvin in some materials.

For $|\Delta| < 1$ the Drude weight at $T = 0$ has been calculated using the BA method [46, 21] and is given by,

$$D_0 = \frac{\pi}{8} \frac{\sin(\pi/\nu)}{\frac{\pi}{\nu}(\pi - \frac{\pi}{\nu})}, \tag{1.34}$$

where $\Delta = \cos(\pi/\nu)$ [3]. For $\Delta > 1$, $D(T = 0) = 0$ as the system is gapped.

At finite temperatures, several numerical and analytical studies indicate that for $|\Delta| < 1$ the spin transport is ballistic [47, 48, 49, 50, 51], in accord with the conjecture that this behavior is related to the integrability of the model [52, 53, 25]. Pursuing this conjecture, one can attempt to use the Mazur inequality eq. (1.14) in order to obtain a bound on the Drude weight and thus establish that the transport is ballistic. Inspection of the known conservation laws for the Heisenberg model [54] shows that already the first nontrivial one, Q_3, has a physical meaning; it corresponds to the energy current, $Q_3 = j^E$ and it can be used to establish a bound for D [26],

$$D(T) \geq \frac{\beta}{2L} \frac{\langle j^z Q_3 \rangle^2}{\langle Q_3^2 \rangle}. \tag{1.35}$$

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[3]The parametrization of Δ in terms of ν is common in the BA literature as the formulation greatly simplifies for $\nu =$ integer.

This expression can be readily evaluated in the high temperature limit ($\beta \to 0$),

$$D(T) \geq \frac{\beta}{2} \frac{8\Delta^2 m^2(1/4 - m^2)}{1 + 8\Delta^2(1/4 + m^2)}, \qquad m = \langle S_l^z \rangle, \qquad (1.36)$$

where m is equal to the magnetization density in the Heisenberg model or to $n-1/2$ in the equivalent fermionic $t-V$ model (n is the density). It establishes that ballistic transport is possible at all finite temperatures in the Heisenberg ($t-V$) model; notice however, that the right hand side vanishes for $m = 0$, that corresponds to the specific case of the antiferromagnetic regime at zero magnetic field or to the $t-V$ model at half-filling. Of course this does not mean that D is indeed zero in these cases as this relation provides only a bound. It should also be remarked that the obtained bound is proportional to Δ^2 and so we do not recover the simple result that $D(T) > 0$ in the XY-limit. Furthermore, it can be shown, using a symmetry argument, that even by taking into account all conservation laws the bound remains zero at $m = 0$ [26].

A BA method based calculation of $D(T)$ for $|\Delta| < 1$ was also performed [55], using a procedure proposed for the Hubbard model [56], that relies upon a certain assumption on the flux dependence (see eq.(1.10)) of bound state excitations ("rigid strings"). The resulting behavior is summarized in Figs. 1.3 and 1.4. From this analysis the following picture emerges:
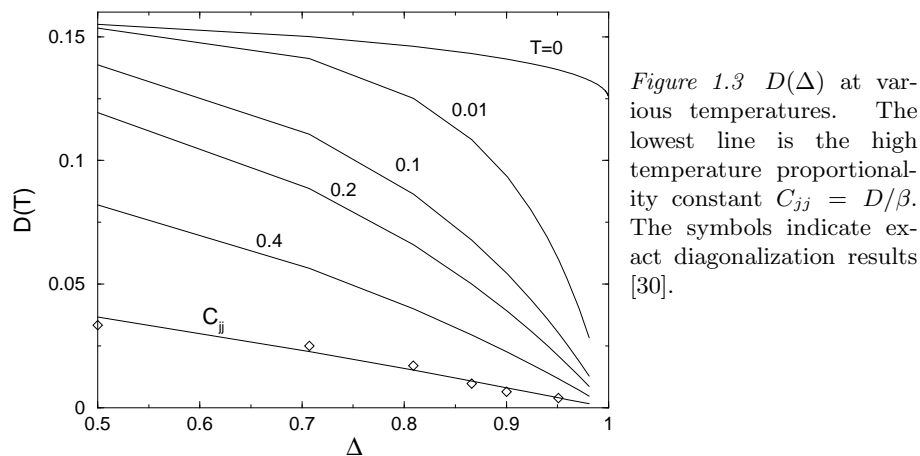

Figure 1.3 $D(\Delta)$ at various temperatures. The lowest line is the high temperature proportionality constant $C_{jj} = D/\beta$. The symbols indicate exact diagonalization results [30].

(i) at zero magnetization, in the easy plane antiferromagnetic regime ($0 < \Delta < 1$), the Drude weight decreases at low temperatures as a power law $D(T) = D_0 - \text{const.}T^\alpha$, $\quad \alpha = 2/(\nu - 1)$;

(ii) in the ferromagnetic regime, $-1 < \Delta < 0$, $D(T)$ decreases quadratically with temperature (as in a noninteracting, XY-system);

(iii) the same low temperature quadratic behavior is true at any finite magnetization;

(iv) for $\beta \to 0$, $D(T) = \beta C_{jj}$ and it can be shown that $D(-\Delta) = D(\Delta)$ by applying a unitary transformation in the expression eq.(1.11);

a closed expression for C_{jj} can be obtained by analytic calculations [57], $C_{jj} = (\pi/\nu - 0.5\sin(2\pi/\nu))/(16\pi/\nu)$ for $|\Delta| < 1$ while $C_{jj} = 0$ for $\Delta > 1$;

(v) at the isotropic antiferromagnetic point ($\Delta = 1$), $D(T)$ seems to vanish, implying non ballistic transport at all finite temperatures.
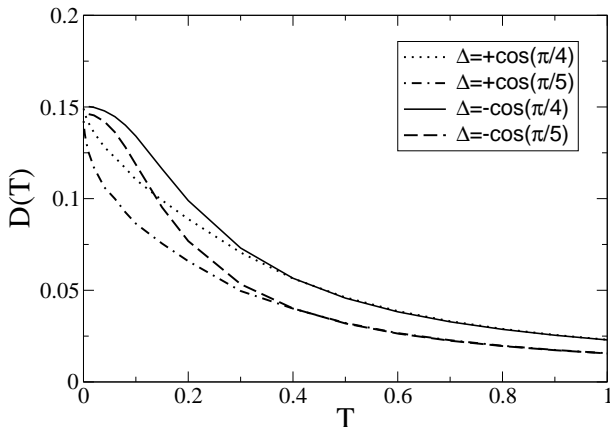

Figure 1.4 Temperature dependence of the Drude weight D vs. T [55]

This last result seems in accord with the most recent NMR data [5]. Of course, the low frequency conductivity must also be examined in order to determine whether there is no anomalous behavior (e.g. power law divergence) that precludes a normal diffusive behavior; such unconventional behavior is presently debated in classical nonlinear 1D systems (see final section of Discussion). It should not be surprising if future rigorous studies reveal that the isotropic Heisenberg exhibits a singular behavior, as it lies at the transition between a gapless and gapped phase.

In this context, we should also mention that the power law decrease of $D(T)$ for $0 < \Delta < 1$ is not corroborated by recent QMC simulations [58]. The disagreement might be due either to the "rigid string" assumption used in the BA analysis or to the very low temperatures, of the order of the energy level spacing, that are studied in the QMC simulations [4].

Considering the limited results obtained so far using the Mazur inequality compared to the exact BA analysis, it remains an open question whether the behavior of the Drude weight can be fully accounted

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[4]Reliable results for the Drude weight can be obtained by QMC simulations only at low temperatures because a sufficiently fine spacing of Matsubara frequencies is required for the extrapolation to zero frequency.

for solely by a proper consideration of conservation laws present in the Heisenberg model.

For $\Delta > 1$ numerical simulations [48] and analytical arguments [51] indicate that the Drude weight vanishes at all temperatures. In this regime, based on ED numerical simulations, it was proposed that a new phase might exist, an "ideal insulator", characterized by vanishing Drude weight and diffusion constant (*dc* conductivity in the fermionic version). This conjecture remains presently still rather tentative, due to the small size of the systems that have been studied so far.

On the other hand, a semiclassical field theory approach [59] concluded that gapped systems are diffusive. This approach is based on a mapping of the massive excitations to impenetrable classical particles of two or more charges (corresponding to different spin directions) that propagate diffusively (see section 5) and it has mostly been used for the analysis of gapped spin-1 systems.

In parallel to these developments, the spin S=1/2 Heisenberg model was studied in the scaling limit using conformal invariance arguments [60, 61]. This field theory approach amounts to considering a linearized spectrum and thus neglecting the effects of curvature, a point that we will further discuss below in section 5. In particular, it was shown that the uniform dynamic susceptibility describes ballistic behavior, the corresponding $1/T_1$ relaxation time was evaluated and the theory was extensively compared to experimental data [62]. Notice, however, that a later experimental NMR work [5] concludes that the $q = 0$ mode of spin transport is ballistic at the $T = 0$ limit, but has a diffusion-like contribution at finite temperatures even for $T << J$. We should remark that, over the years, the most common interpretation of NMR experiments was within the diffusion phenomenology, as for instance for the $S = 5/2$ TMMC compound [63].

Finally, the finite (q, ω) response functions of the S=1/2 model at $T = 0$ were studied by the bosonization technique [64] after mapping it to spinless fermions (eq.(1.19)). For $\Delta < 1$, the conductivity shows the typical ballistic form; for $\Delta > 1$ it vanishes below the gap, showing a square-root frequency dependence above.

Turning now to energy transport, it is easy to see that the energy current is a conserved quantity [65, 26] for all values of the anisotropy Δ implying that the currents do not decay and so the thermal conductivity is infinite. This peculiarity has also been noticed by an earlier analysis of moments at infinite temperature [66]. So the quantity characterizing thermal transport is the equal time correlation $\langle j^E j^E \rangle$ that represents
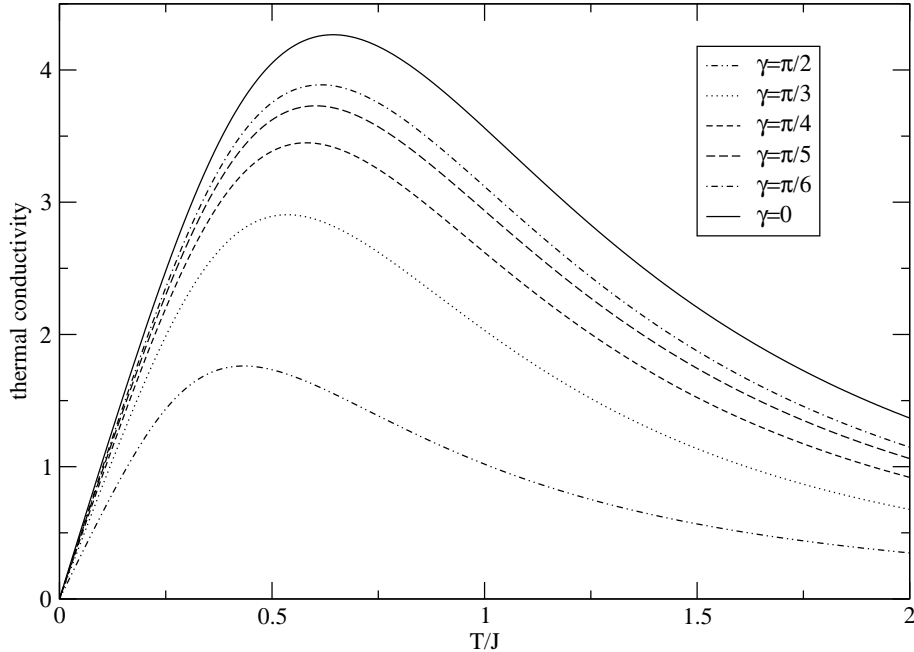
Figure 1.5. Thermal conductivity, $\langle j^E j^E \rangle$ in units of J^2, for various anisotropy parameters $\Delta = \cos(\gamma)$ [67].

the weight under the low frequency peak that will develop from the zero frequency $\delta-$function when a dissipative mechanism is introduced. This picture is analogous to that of the electrical conductivity illustrated in Fig. 1.1. It implies that, given an estimate of the temperature dependence of the characteristic scattering time one is able to extract the value of the *dc* thermal conductivity, further assuming some form (e.g. eq.(1.8)) for the low frequency behavior.

This quantity has also recently been exactly calculated using the BA method [67]; it is shown in Fig. 1.5.

Furthermore, the experimental observation of unusually high thermal conductivity in ladder compounds [1] motivated the theoretical study of the thermal Drude weight in 1D anisotropic, frustrated and ladder spin-1/2 systems [68, 69]; the proposal of unconventional thermal transport in these systems is still debated.

Finally, the S=1 Heisenberg chain shows a qualitatively different behavior characterized by the presence of an energy (Haldane) gap at low energies. The S=1 Heisenberg model is not integrable but the physics at low energies is usually mapped onto the quantum nonlinear-σ model

that is again an integrable system. The results known on this model will be briefly discussed in section 5 along with a semiclassical approach to describe this type of gapped systems. The same low energy mapping is used for the analysis of "ladder" compounds.

As a guide to experimental investigations and theoretical studies, we can recapitulate the above discussion of the dynamics of the Heisenberg S=1/2 model as follows. It seems clear that ballistic behavior at all temperatures should be expected in the easy-plane regime and at all finite magnetizations, while the isotropic point is a subtle borderline case. The behavior in the easy-axis antiferromagnetic regime might be particularly interesting and it is not settled at the moment. Exceptionally high thermal conductivity should be expected in all regimes.

To complete the above picture, we should stress that not much is known on the low frequency behavior of the conductivities at finite temperature. This leaves open the possibility of unconventional behavior, neither ballistic nor simple diffusive but one characterized by long time tails, giving rise to power law (or logarithmic) behavior at low frequencies.

4. Hubbard model

The prototype model for the description of electron-electron correlations is the Hubbard model given by the Hamiltonian,

$$H = \sum_l h_l = (-t) \sum_{\sigma,l} (c_{l\sigma}^\dagger c_{l+1\sigma} + h.c.) + U \sum_l n_{l\uparrow} n_{l\downarrow} \qquad (1.37)$$

where $c_{l\sigma}(c_{l\sigma}^\dagger)$ are annihilation (creation) operators of fermions with spin $\sigma = \uparrow, \downarrow$ at site l and $n_{l\sigma} = c_{l\sigma}^\dagger c_{l\sigma}$.

At half-filling (n=1, 1 fermion per site) it describes a Mott-Hubbard insulator for any value of the repulsive interaction $U > 0$, while it is a metal at any other filling.

The one dimensional Hubbard model is also integrable by the Bethe ansatz method and its phase diagram, elementary excitations, correlation functions have been extensively studied [10, 70].

4.1 Currents

Similarly to the Heisenberg model, we can discuss the electrical, spin and thermal conductivity by defining the charge j, spin j^s and energy j^E currents from the respective continuity equations of the local particle density n_l,

$$\frac{\partial n_l}{\partial t} + \nabla j_l = 0, \quad j = \sum_l j_l = \sum_{l\sigma} j_{l\sigma} = (-t)\sum_{\sigma,l}(ic_{l\sigma}^\dagger c_{l+1\sigma} + h.c.), \quad (1.38)$$

spin density $n_{l\uparrow} - n_{l\downarrow}$,

$$\frac{\partial(n_{l\uparrow} - n_{l\downarrow})}{\partial t} + \nabla j_l^s = 0, \quad j^s = \sum_l j_l^s = \sum_l j_{l\uparrow} - j_{l\downarrow} \qquad (1.39)$$

and energy density h_l,

$$\frac{\partial h_l}{\partial t} + \nabla j_l^E = 0, \quad j^E = \sum_{l,\sigma} j_{l\sigma}^E \qquad (1.40)$$

$$j_{l\sigma}^E = (-t)^2(ic_{l+1\sigma}^\dagger c_{l-1\sigma} + h.c.) - \frac{U}{2}j_{l,\sigma}(n_{l,-\sigma} + n_{l+1,-\sigma} - 1).$$

4.2 Electrical and thermal transport

With respect to the electrical conductivity the interaction U and density dependence of the Drude weight D at zero temperature has been established using the BA method [71, 72, 73] (see Fig. 1.6). There are two simple limits:

(i) The free fermion case $U = 0$ where j is conserved and $D_0 = \frac{2t}{\pi}\sin\frac{\pi n}{2}$ where n is the density of fermions $(n = 2k_F/\pi)$. Here D_0 vanishes for an empty band $n = 0$ and a filled band $n = 2$, being maximum at half filling, $n = 1$.

(ii) Another simple limit is $U = \infty$. Since in this case the double occupation of sites is forbidden, fermions behave effectively as spinless fermions and the result is $D_0 = \frac{t}{\pi}|\sin(\pi n)|$; here D_0 vanishes also at half filling.

Analytical results in 1D indicate that the $D_0 = 0$ value at half filling persists in the Hubbard model for all $U > 0$, whereby the density dependence $D(n)$ is between the limits $U = 0$ and $U = \infty$. The insulating state at half filling is a generic feature of a wider class of 1D models characterized by repulsive interactions, such as the t-V model (discussed above), the t-J model etc.

In Fig. 1.6, along with the Drude weight, the zero temperature (ballistic) Hall constant R_H of a quasi-1D system is also shown. According to a recent formulation [74], R_H can be expressed in terms of the derivative of the Drude weight with respect to the density,

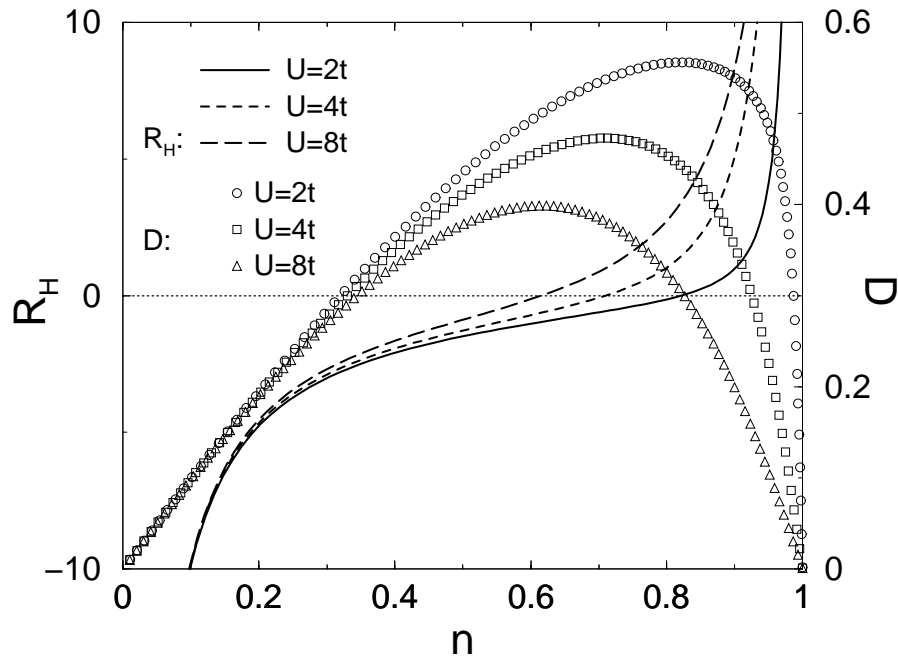$$R_H = -\frac{1}{D}\frac{\partial D}{\partial n}. \qquad (1.41)$$

Figure 1.6. Drude weight D and R_H for the quasi-1D Hubbard model from expression (1.41).

The Hall constant is the classical way for determining the sign of the charge carriers. For a strictly one dimensional system of course it makes no sense to discuss the Hall effect; but if we consider a quasi-one dimensional system with interchain coupling characterized by a hopping $t' \to 0$, then within this formulation we recover a simple picture for the behavior of the sign of carriers as a function of interaction. In agreement with intuitive semiclassical arguments, the Hall constant behaves as $R_H \simeq -1/n$ at low densities changing to $R_H \simeq +1/\delta(\delta = 1 - n)$ near half-filling, with the turning point depending on the strength of the interaction U. Notice that if $D \propto n$ with a small proportionality constant, that would be interpreted within a single particle picture as indicative of a large effective mass, then we would still find $R_H \simeq -1/n$. This observation might be relevant in the context of recent optical and Hall experiments [75, 7] where a small Drude weight is observed although the Hall constant indicates a carrier density of order of one.

Recently, using the form factor and DMRG methods the frequency dependence of the conductivity at half filling and at $T = 0$ has also been
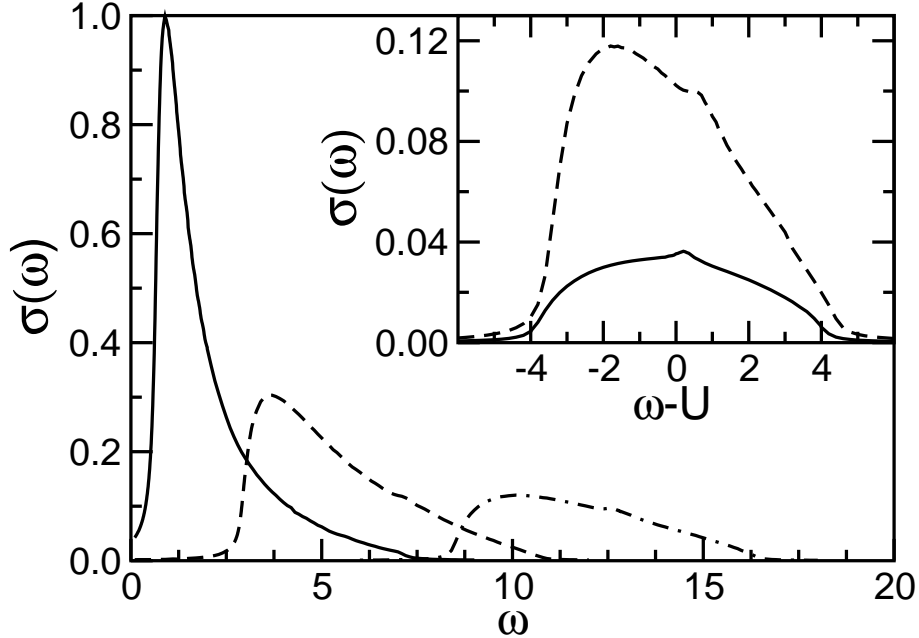
Figure 1.7. Optical conductivity at $T = 0$ for $U/t = 3, 6, 12$ (from left to right) calculated with DDMRG on a 128-site lattice [76]. Inset: $\sigma(\omega)$ for $U/t = 12$ (dashed) and 40 (solid) calculated on a 64-site chain.

studied [76] and is shown in Fig. 1.7. The DMRG method provided the entire absorption spectrum for all but very small couplings where the field theoretical approach was used; the two methods are in excellent agreement in their common regime of applicability. As expected, the Drude weight is zero, signaling an insulating state (for a detailed analysis of the scaling of D with system size at and close to half-filling, see [77]) and the finite frequency conductivity vanishes up to the gap. Above the gap, a square root dependence is observed but not a divergence; this behavior is in contrast to that obtained by the Luttinger liquid method [78] and it is typical of a Peierls (band) insulator where a divergence occurs. This absence of a singularity is also in agreement with a rigorous analysis of the sine-Gordon (sG) field theory (see section 5), the generic low energy effective model for the description of a Mott-Hubbard insulator.

To complete the zero temperature picture, the frequency dependent conductivity of the Hubbard model out of half-filling has been studied using results from the BA method and symmetries [79]. A broad ab-

sorption band was found separated from the Drude peak at $\omega = 0$ by a pseudogap; this pseudo-gap behavior is in contrast to the ω^3 dependence found within the Luttinger liquid analysis [78].

Again, at all finite temperatures the transport is ballistic characterized by a finite Drude weight. In an identical way to the Heisenberg model, this can easily be established by the Mazur inequality using the first nontrivial conservation law Q_3. For the Hubbard model Q_3 differs from the energy current j^E by the replacement of U by $U/2$ [26]. Evaluating $\langle jQ_3 \rangle^2 / \langle Q_3^2 \rangle$ for $\beta \to 0$ we obtain,

$$D(T) \geq \frac{\beta}{2L} \frac{\langle jQ_3 \rangle^2}{\langle Q_3^2 \rangle} = \frac{\beta}{2} \frac{[U \sum_\sigma 2n_\sigma (1 - n_\sigma)(2n_{-\sigma} - 1)]^2}{\sum_\sigma 2n_\sigma (1 - n_\sigma)[1 + U^2(2n_{-\sigma}^2 - 2n_{-\sigma} + 1)]}, \tag{1.42}$$

where n_σ are the densities of $\sigma = \uparrow, \downarrow$ fermions.

By inspection we can again see that from this inequality we cannot obtain a finite bound for $D(T)$ for $n_\uparrow + n_\downarrow = 1$. Nevertheless, a full BA calculation [56] seems to show that the Drude weight at half-filling is exponentially activated $D(T) \sim \frac{1}{\sqrt{T}} e^{-E_{gap}/T}$ at low temperatures and decreases as T^2 out of half-filling. Thus the zero temperature insulator turns to an ideal conductor at finite temperatures. Notice that this behavior is different from the one in the Heisenberg (or "t-V") model in the gapped phase ($\Delta > 1$) where the Drude weight seems to vanish at all finite temperatures. We can conjecture that this distinct behavior of insulating phases can be understood in the framework of the corresponding low energy sine-Gordon field theory as these two models map to different parameter regimes of the sG model [64, 13]. A very similar calculation, using the Mazur inequality, can also be carried out for the long time asymptotics of the spin current, j^s, correlations. It gives a finite bound, and thus ballistic spin transport for $n_\uparrow - n_\downarrow \neq 0$; no BA calculation has so far been performed for the spin conductivity.

On the thermal conductivity we find similar results, namely a finite value on the long time decay of energy current correlations, which can readily be evaluated for $\beta \to 0$ [26],

$$\lim_{t \to \infty} \langle j^E(t) j^E \rangle = C_{j^E j^E} \geq \frac{\langle j^E Q_3 \rangle^2}{\langle Q_3^2 \rangle}. \tag{1.43}$$

Again this inequality gives a finite bound for a system out of half-filling as long as $n_\uparrow + n_\downarrow \neq 0$ and this for any magnetization. For this model the

actual temperature dependence of $C_{j^E j^E} = \lim_{t\to\infty}\langle j^E(t)j^E\rangle = C_{j^E j^E}$ has not yet been evaluated. Finally, the low temperature thermoelectric power was studied using the Bethe ansatz picture for the charge (holons) and spin (spinons) excitations [80]. The resulting sign of the thermopower close to the Mott-Hubbard insulating phase is consistent with the one derived from the Hall constant above, $S \sim sign(1-n)T|m^*|/|1-n|^2$.

In summary, we have shown that the prototype model for describing electron correlations in one dimensional systems, the Hubbard model, shows unconventional, ballistic charge, spin and thermal transport at all finite temperatures. Of course real quasi-one dimensional materials are presumably characterized by longer range than the Hubbard U interactions. So, although the above picture should be taken into account in the interpretation of experiments, (quasi-) one dimensional magnetic compounds might presently appear as better candidates for the experimental observation of these effects. Theoretically, the full frequency dependence of the conductivities at finite temperatures remains to be established.

5. Effective field theories

An alternative to analyzing the transport of quasi-one dimensional materials within microscopic models, as described in previous sections, is to approach the problem within effective low energy models for interacting electrons, i.e. starting with the Luttinger liquid Hamiltonian. This path is very attractive since it represents the counterpart of the usual Landau phenomenological approach to Fermi liquid in higher-D electronic systems. It should be pointed out that even in a 3D system the continuum field theory is not enough to describe a current decay and Umklapp processes are finally responsible for a finite intrinsic resistivity $\rho(T) \propto T^2$ [81].

In an effective (low energy) field theoretical model for 1D interacting electrons the band dispersion around the Fermi momenta $k = \pm k_F$ is linearized and left- and right- moving excitations are defined. Apart from Umklapp terms, the model of interacting fermions can then be mapped onto the well known Luttinger liquid Hamiltonian [16, 13] and analyzed via the bosonization representation. In particular one obtains for the charge sector,

$$H_0 = \frac{1}{2\pi} \int dx \Big[u_\rho K_\rho (\pi\Pi_\rho)^2 + \frac{u_\rho}{K_\rho}(\partial_x\phi_\rho)^2 \Big], \qquad (1.44)$$

where the charge density is $\rho(x) = \partial_x\phi_\rho$ and Π_ρ is the conjugate momentum to ϕ_ρ. Interactions appear only via the velocity parameter u_ρ and

Luttinger exponent K_ρ. The charge current in such a Luttinger model, $j = \sqrt{2}u_\rho K_\rho \Pi_\rho$, is clearly conserved in the absence of additional terms.

Umklapp terms can as well be represented with boson operators,

$$H_{\frac{1}{2m}} = g_{\frac{1}{2m}} \int dx \cos(m\sqrt{8}\phi_\rho(x) + \delta x), \qquad (1.45)$$

where m is the commensurability parameter ($m = 1$ at half-filling - one particle per site, $m = 2$ for quarter filling - one particle for two sites etc) and δ the doping deviation from the commensurate filling. In principle, the mapping of a particular (tight binding) microscopic model onto a field theory model, e.g. via perturbation theory, generates terms H_m with arbitrary m. While Umklapp terms are irrelevant in the sense of universal scaling of the static properties, they appear to be crucial for transport. They drive a metal at half-filling to an insulator, while at an arbitrary (incommensurate) filling they should cause a finite resistivity since the current is not conserved any more (for an overview of the transport properties emerging within the Luttinger liquid picture see [82]).

However, the proper treatment of transport within the Luttinger picture in the presence of Umklapp processes is quite involved and even controversial. Giamarchi [78] first calculated the effect of Umklapp scattering within lowest order perturbation theory for the memory function $M(\omega)$; he thus determined the low-ω behavior of the dynamical conductivity $\sigma(\omega) \propto 1/(\omega + M(\omega))$ that yielded a nonzero finite temperature conductivity. At the same time he realized, by using the Luther-Emery method, that the Umklapp term can be absorbed in the Hamiltonian in such a way as to conserve the current and pointed out the possibility of infinite dc conductivity even in the presence of Umklapp. A similar lowest-order analysis [83] for general commensurate filling predicts at $T = 0$ that $\sigma(\omega) \propto \omega^{\nu-2}$ and the resistivity $\rho(T) \propto T^\nu$ with $\nu = 4n^2 K_\rho - 3$. On the other hand, Rosch and Andrei [84] pointed out that even in the presence of general Umklapp terms there exist particular operators, linear combinations of the translation operator and number difference between left- and right- moving electrons, which are conserved. Since in general such operators have a nonvanishing overlap with the current operator j, this leads to finite $D(T > 0) > 0$ if only one Umklapp term is considered. At least the interplay of two noncommuting Umklapp processes is needed to yield a finite resistivity $\rho(T > 0) > 0$.

From a different perspective Ogata and Anderson [85] argued that because of spin-charge separation in 1D systems an effect analogous to phonon drag (in this case spinon-holon drag) appears that leads to a fi-

nite dissipation. Using a Landauer like semi-phenomenological approach they concluded the existence of a linear-T resistivity and linear frequency dependence of the optical conductivity.

The bosonization of the Luttinger liquid model leads [13] to the quantum sine-Gordon model (eq.(1.45)) which is an integrable system and has extensively been studied as a prototype nonlinear quantum (or classical) field theory. It is the generic field theory for describing the low energy properties of one dimensional Mott insulators. The thermodynamic properties and excitation spectrum consisting of solitons/antisolitons and breather states have been established by semiclassical and BA techniques [10]. Presently, there is an effort to determine the transport properties of this model rigorously. In particular, the frequency dependence of the zero temperature conductivity in the commensurate (insulating) phase, zero soliton sector, has been evaluated using the form factor approach [86]. The main result is that the square root singularity at the optical gap, characteristic of band insulators, is generally absent and appears only at the Luther-Emery point; furthermore, the perturbative result [78] is recovered only at relatively high frequencies. Besides these studies, the Drude weight and optical response near the metal-insulator transition, in the incommensurate phase at zero temperature, have also been studied by Bethe ansatz [87] and semiclassical methods [88]. Still, a rigorous evaluation of the Drude weight and frequency dependence of the conductivity at finite temperatures is missing; nevertheless, we can plausibly argue that because of the integrability of the sine-Gordon model, it will turn out that also this model describes an ideal conductor at least over some interaction range. Thus, it might remain an open question which scattering processes and/or band curvature must be taken into account in order to recover a normal, diffusive behavior at finite temperatures.

Finally, it is well known [89] that the spectrum of integer spin and even-leg ladder systems is gapped and that the low energy physics is described by the one-dimensional quantum $O(3)$ nonlinear sigma model [40].

In imaginary time τ the action at inverse temperature β is given by

$$S = \frac{c}{2g} \int_0^\beta d\tau \Big[(\partial_x n_\alpha)^2 + \frac{1}{c^2} (\partial_\tau n_\alpha)^2 \Big], \qquad (1.46)$$

where x is the spatial coordinate, c a characteristic velocity, $\alpha = 1, 2, 3$ is an $O(3)$ vector index and $n_\alpha(x, \tau)$ a unit vector field $n_\alpha^2(x, \tau) = 1$.

In a series of works, Sachdev and collaborators [90, 91, 92] developed a picture of the low and intermediate temperature spin dynamics based on

the idea that the spin excitations can be mapped to an integrable model describing a classical gas of impenetrable particles (of a certain number of species depending on the spin), a problem that can be treated analytically. Within this framework they have extensively analyzed NMR experiments on S=1 compounds [93] and they concluded that these systems behave diffusively. In contrast to this semiclassical approach, using the Bethe ansatz solution of the quantum nonlinear$-\sigma$ model [94], Fujimoto [95] found a finite Drude weight, exponentially activated with temperature, and he thus concluded that the spin transport at finite temperatures is ballistic. The origin of this discrepancy is not clear at the moment and can be due either to a subtle role of quantum effects on the dynamics that is neglected in the semiclassical approach or to a particular limiting procedure (the magnetic field going to zero) in the BA solution.

6.   Discussion

We hope that the above presentation demonstrated that the transport theory of one dimensional quantum systems is a rapidly progressing field, fueled by both theoretical and experimental developments. Still, on the question, what is the finite temperature conductivity of bulk electronic or magnetic systems described by strongly interacting one dimensional Hamiltonians, it is fair to say that no definite answer has so far emerged nor there is a clear picture of the relevant scattering mechanisms.

In this context, it is interesting and instructive to draw an analogy with the development of the respective field in classical physics, namely the finite temperature transport in one dimensional nonlinear systems. Interestingly, in this domain we are also witnessing a flurry of activity after several decades of studies. Again, the issue of ballistic versus diffusive (usually energy) transport in a variety of models and the necessary ingredients for observing normal behavior is sharply debated [96, 97]. Similarly to the quantum systems, numerical simulations are intensely employed along with analytical approaches and discussions on the conceptual foundations of transport theory.

For quantum systems it is reasonable to expect that the finite temperature transport properties of integrable models will, in the near future, be amenable to rigorous analysis by mathematical techniques, for instance in the framework of the Bethe ansatz method. At the same time, as we mentioned earlier, it is amusing to notice that the integrable systems that we can exactly analyze, present singular transport properties presumably exactly because of their integrability.

To obtain normal behavior, it is reasonable to invoke perturbations destroying the integrability of the model, as for instance longer range interactions, interchain coupling, coupling to phonons, disorder etc. In this scenario, it is then necessary to find ways to study the effect of perturbations around an integrable system and in particular to determine the vicinity in parameter space around the singular-integrable point where unconventional transport can be detected. This issue is also extensively studied in classical systems as it is the most relevant in the interpretation of experiments and in estimating the prospects for technological realizations. It is worth keeping in mind the possibility that integrable interactions actually render the system more immune to perturbations, an effect well known and exploited in classical nonlinear systems [9].

Related to this line of argument is the following question. If integrable models show ballistic transport and low energy effective theories like the sine-Gordon model are also integrable, then which mechanisms are necessary to obtain dissipative behavior ?

Of course it is also possible that the conventional picture according to which only integrable systems show ballistic transport might well be challenged. One dimensional nonintegrable quantum systems could also show singular transport in the form either of a finite Drude weight or low frequency anomalies. This behavior has been observed in classical nonintegrable nonlinear systems where the current correlations decay to zero in the long time limit but too slowly, so that the integral over time (giving the dc conductivity) diverges. The opposite behavior might also be realized, namely that integrable quantum systems show normal diffusive transport in some region of interaction parameter space (this possibility was raised in the case of gapped systems as the easy-axis spin $1/2$ Heisenberg model or the quantum nonlinear$-\sigma$ model, see section 5). Furthermore, the issue of the crossover of the dynamics between quantum and classical systems has, at the moment, very little been explored and in particular the question whether quantum fluctuations might stabilize ballistic transport behavior.

To address all the above open issues there is a clear need for the development of reliable analytical and numerical simulation techniques (as the DMRG or QMC) to tackle the evaluation of dynamic correlations at low temperatures. In particular, progress is needed to include the coupling between the different, magnetic, electronic and phononic, excitations.

In summary, one of the most fascinating aspects in this field is to understand the extent to which the so successful physics, experimental and technological realizations of classical (integrable) nonlinear systems

can be carried over to the quantum world of many body (quasi-) one dimensional electronic or magnetic strongly interacting systems. This effort is accompanied by the experimental challenge to synthesize novel materials/systems that realize this physics.

Acknowledgments

References

[1] "Quantum effects in heat transport in one-dimensional systems", H.R. Ott, Chapter 10 of present volume.

[2] A.V. Sologubenko et al., Phys. Rev. Lett. **84**, 2714 (2000); A. V. Sologubenko et al., Phys. Rev. B**64**, 054412 (2001).

[3] C. Hess et al., Phys. Rev. B**64**, 184305 (2001).

[4] M. Takigawa et al., Phys. Rev. Lett. **76**, 4612 (1996).

[5] K.R. Thurber et al., Phys. Rev. Lett. **87**, 247202 (2001).

[6] M. Dressel, A. Schwartz, G. Grüner and L. Degiorgi, Phys. Rev. Lett. **77**, 398 (1996).

[7] "Electrodynamic response in "one-dimensional" chains", L. Degiorgi, Chapter 8 of present volume.

[8] F.D.M. Haldane, J. Phys. C**14**, 2585 (1981).

[9] "Optical solitons: theoretical challenges and industrial perspectives", editors: V.E. Zakharov and S. Wabnitz, Les Houches Workshop, Springer (1998).

[10] "Quantum Inverse Scattering Method and Correlation Functions", V.E. Korepin, N.M. Bogoliubov and A.G. Izergin, Cambridge Univ. Press (1993).

[11] H.G. Evertz, Adv. Phys. **52**, 1 (2003).

[12] "Density-matrix renormalization : a new numerical method in physics", Workshop Proceedings, Dresden, I. Peschel et al. eds., Springer (1999).

[13] V.J. Emery in *Highly Conducting One-dimensional Solids*, ed. by J. Devreese *et al.* (Plenum, New York, 1979), p.247.

32

[14] J. Solyom, Adv. Phys. **28**, 209 (1979).

[15] J. Voit, Rep. Prog. Phys. **58**, 977 (1995)

[16] "Luttinger liquids: the basic concepts", K. Schönhammer, Chapter 9 of present volume.

[17] see also "Optics of correlated systems", A.J. Millis, Chapter 6 of present volume.

[18] W. Kohn, Phys. Rev. **133**, A171 (1964).

[19] P.F. Maldague, Phys. Rev. B**16**, 2437 (1977).

[20] D. Baeriswyl, C. Gros and T.M. Rice, Phys. Rev. B**35**, 8391 (1987).

[21] B.S. Shastry and B. Sutherland, Phys. Rev. Lett. **65**, 243 (1990).

[22] J.T. Edwards and D.J. Thouless, J. Phys. C**5**, 807 (1972).

[23] M. Imada, A. Fujimori and Y. Tokura, Rev. Mod. Phys. **70**, 1039 (1998).

[24] J. Jaklič and P. Prelovšek, Adv. Phys. **49**, 1 (2000).

[25] H. Castella, X. Zotos, P. Prelovšek, Phys. Rev. Lett. **74**, 972 (1995).

[26] X. Zotos, F. Naef and P. Prelovšek, Phys. Rev. B**55** 11029 (1997).

[27] P. Mazur, Physica **43**, 533 (1969).

[28] M. Wilkinson, Phys. Rev. B**41**, 4645 (1990).

[29] B.D. Simons and B.L. Altshuler, Phys. Rev. Lett. **70** 4063 (1993).

[30] F. Naef and X. Zotos, J. Phys. C. **10**, L183 (1998); F. Naef, Ph. D. thesis no.2127, EPF-Lausanne (2000).

[31] W. Götze and P. Wölfle, Phys. Rev. B**6**, 1226 (1972).

[32] A.G. Redfield and W.N. Yu, Phys. Rev. **169**, 443 (1968); erratum, Phys. Rev. **177**, 1018 (1969).

[33] J. Jaklič, P. Prelovšek, Phys. Rev. B**49**, 5065 (1994).

[34] M.W. Long, P. Prelovšek, S. El Shawish, J. Karadamoglou and X. Zotos, cond-mat/0302211.

[35] M. Jarrel and J.E. Gubernatis, Phys. Rep. **269**, 135 (1996).

[36] T.D. Kühner and S.R. White, Phys. Rev. B**60**, 335 (1999).

[37] E. Jeckelmann, Phys. Rev. B**66**, 045114 (2002).

[38] F. Naef, X. Wang, X. Zotos, W. von der Linden, Phys. Rev. **B60**, 359 (1999).

[39] E. Lieb, T. Schultz and D. Mattis, Ann. Phys. (N.Y.) **16**, 407 (1961).

[40] F.D.M. Haldane, Phys. Lett. **93A**, 464 (1983).

[41] L.P. Kadanoff and P.C. Martin, Ann. Phys. **24**, 419 (1963).

[42] N.A. Lurie, D.L. Huber and M. Blume, Phys. Rev. **B9**, 2171 (1974).

[43] O. F. de Alcantara Bonfim and G. Reiter, Phys. Rev. Lett. **69**, 367 (1992).

[44] G. Müller, Phys. Rev. Lett. **60**, 2785 (1988); N. Srivastava, J-M. Liu, V.S. Viswanath and G. Müller, J. Appl. Phys. **75**, 6751 (1994) and references therein.

[45] T. Niemeijer, Physica **36**, 377 (1967).

[46] F.D.M. Haldane, Phys. Rev. Lett. **45**, 1358 (1980).

[47] K. Fabricius and B.M. McCoy, Phys. Rev. B**57**, 8340 (1998) and references therein.

[48] X. Zotos and P. Prelovšek, Phys. Rev. B**53**, 983 (1996).

[49] B.N. Narozhny, A.J. Millis and N. Andrei, Phys. Rev. B**58**, 2921 (1998).

[50] B.N. Narozhny, Phys. Rev. B**54**, 3311 (1996).

[51] N.M.R. Peres, P.D. Sacramento, D.K. Campbell and J.M.P. Carmelo, Phys. Rev. B**59**, 7382 (1999).

[52] B.M. McCoy, in *Statistical Mechanics and Field Theory*, ed. V.V. Bazhanov and C.J. Burden (World Scientific), 26 (1995).

[53] K. Saito, S. Takesue and S. Miyashita, Phys. Rev. E**54**, 2404 (1996).

[54] M. P. Grabowski and P. Mathieu, Ann. Phys. **243**, 299 (1996).

[55] X. Zotos, Phys. Rev. Lett. **82**, 1764 (1999).

[56] S. Fujimoto and N. Kawakami, J. Phys. A**31**, 465 (1998).

[57] A. Klümper, private communication.

[58] J. V. Alvarez and C. Gros, Phys. Rev. Lett. **88**, 077203 (2002).

[59] K. Damle and S. Sachdev, Phys. Rev. B**57**, 8307 (1998).

[60] H.J. Schulz, Phys. Rev. B**34**, 6372 (1986).

[61] S. Sachdev, Phys. Rev. B**50**, 13006 (1994).

[62] M. Takigawa, O.A. Starykh, A.W. Sandvik and R.R.P. Singh, Phys. Rev. B**56**, 13681 (1997) and references therein.

[63] D. Hone, C. Scherer ad F. Borsa, Phys. Rev. B**9**, 965 (1974).

[64] R. Shankar, Int. J. Mod. Phys. B**4**, 2371 (1990).

[65] T. Niemeijer and H.A.W. van Vianen, Phys. Lett. **34A**, 401 (1971).

[66] D.L. Huber and J.S. Semura, Phys. Rev. B**182**, 602 (1969).

[67] A. Klümper and K. Sakai, J. Phys. A. **35**, 2173 (2002).

[68] J. V. Alvarez and C. Gros, Phys. Rev. Lett. **89**, 156603 (2002).

[69] F. Heinrich-Meisner, A. Honecker, D.C. Cabra and W. Brenig, Phys. Rev. B**66**, 140406 (2002).

[70] T. Deguchi, F.H.L. Essler, F. Göhmann, A. Klümper, V.E. Korepin, K. Kusakabe, Phys. Rep. **331** (5), 197 (2000).

[71] F.D.M. Haldane, Phys. Lett. **81A**, 153 (1981).

[72] H.J. Schulz, Phys. Rev. Lett. **64**, 2831 (1990).

[73] N. Kawakami and S.K. Yang, Phys. Rev. B**44**, 7844 (1991).

[74] X. Zotos, F. Naef, M. Long, and P. Prelovšek, Phys. Rev. Lett. **85**, 377 (2000).

[75] G. Mihaly et al., Phys. Rev. Lett. **84**, 2670 (2000).

[76] E. Jeckelmann, F. Gebhard and F.H.L. Essler, Phys. Rev. Lett. **85**, 3910 (2000).

[77] C.A. Stafford and A.J. Millis, Phys. Rev. B**48**, 1409 (1993).

[78] T. Giamarchi, Phys. Rev. B **44**, 2905 (1991); Phys. Rev. B **46**, 342 (1992).

[79] J.M.P. Carmelo, N.M.R. Peres and P.D. Sacramento, Phys. Rev. Lett. **84**, 4673 (2000); J. M. P. Carmelo, J. M. E. Guerra, J. M. B. Lopes dos Santos, and A. H. Castro Neto, Phys. Rev. Lett. 83, 3892 (1999)

[80] C.A. Stafford, Phys. Rev. B**48**, 8430 (1993).

[81] "Fundamentals of the Theory of Metals", A.A. Abrikosov, North-Holland (1988).

[82] T. Giamarchi, Physica B **230-232**, 975 (1997).

[83] T. Giamarchi and A. Millis, Phys. Rev. B**46**, 9325 (1992).

[84] A. Rosch and N. Andrei, Phys. Rev. Lett. **85**, 1092 (2000).

[85] M. Ogata, P.W. Anderson, Phys. Rev. Lett. **70**, 3087 (1993).

[86] D. Controzzi, F.H.L. Essler and A.M. Tsvelik, Phys. Rev. Lett. **86**, 680 (2001).

[87] E. Papa and A.M. Tsvelik, Phys. Rev. B**63**, 085109 (2001).

[88] D.N. Aristov, V.V. Cheianov and A. Luther, Phys. Rev. B**66**, 073105 (2002).

[89] E. Dagotto and T.M. Rice, Science **271**, 618 (1996).

[90] S. Sachdev and K. Damle, Phys. Rev. Lett. **78**, 943 (1997); K. Damle and S. Sachdev, Phys. Rev. B**57**, 8307 (1998).

[91] "Quantum Phase transitions", S. Sachdev, Cambridge University Press (1999).

[92] C. Buragohain and S. Sachdev, Phys. Rev. B**59**, 9285 (1999).

[93] M. Takigawa et al., Phys. Rev. Lett. **76**, 2173 (1996).

[94] P.B. Wiegmann, Phys. Lett. **152B** 209 (1985); JETP Lett. **41** 95 (1985).

[95] S. Fujimoto, J. Phys. Soc. Jpn., **68**, 2810 (1999).

[96] F. Bonetto, J.L. Lebowitz and L. Rey-Bellet, "Mathematical Physics 2000", 128, Imp. Coll. Press, London (2000).

[97] S. Lepri, R. Livi and A. Politi, Phys. Rep. **377**, 1 (2003).