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Supersymmetry—Extended!

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Abstract: Since the circumnavigation of the Coleman-Mandula theorem via graded algebras in the early Seventies, supersymmetry has become a ubiquitous fixture of theoretical high energy physics. Phenomenologically, the most interesting supersymmetric extensions of the Standard Model are generated by a single spinorial charge in a four dimensional spacetime. While this restriction is the only way to obtain the chiral structure of the Standard Model, it is certainly not the unique realization of the super-Poincare algebra. Recent developments in string theory (including those centered on duality) emphasize the importance of supersymmetric theories with $\mathcal{N} > 1$ supercharges, in $d \neq 4$ dimensions. In the following review, we develop the formal structure of these "extended" supersymmyetric theories in various spacetime dimensions.

Contents

1. What's Super about SUSY?

The study of field theories with supersymmetry (SUSY) is far more than an exercise in superlatives. Identified by Haag, Lopuszanski, and Sohnius as the unique circumnavigation of the Coleman-Mandula theorem in the mid Seventies [1], SUSY has since secured a special place in the hearts of many theoreticians. This despite the fact that high energy physicists have, in the Standard Model, a perfectly viable framework that makes accurate and testable predictions without any mention of "sparticles".

In many ways, the Standard Model is in its autumn years. The next generation of high energy particle physics experiments have already begun taking data at the Large Hadron Collider beneath CERN, and physicists the world over are crossing their fingers for "something new". As the experimentally accessible energy scale progresses into the TeV range, physics unexplained by the Standard Model (such as the infamous hierarchy problem) becomes increasingly relevant. Supersymmetry is far and away one of the most popular of the "somethings new"'s—it can ameliorate the hierarchy problem, unify the Standard Model gauge couplings, and appears as a consequence of consistent theories of quantum gravity. All of these topics are aesthetically and scientifically fascinating, and yet none of them will be addressed in this review. Instead, we will depart from the most practical implementation of SUSY, the so called $\mathcal{N} = 1$ theory, and spend our time developing the seemingly esoteric theories of multiple spinor supercharges and extra dimensions.

In some important ways, the utility of SUSY is independent of its realization in nature. Advances in the study of strings (which are necessarily supersymmetric theories in ten dimensions if they are to be phenomenologically viable) have uncovered an elegant assortment of dualities that unite seemingly disparate theories. These theories (like [2]) live in different dimensions, have very different particle content, and carry super Poincaré algebras generated by more than one spinor supercharge. Yet somehow they often find themselves in the universality class of real life. The goal of this review is simple—we aim for familiarity with the formal structure of supersymmetry with $\mathcal{N}$ = anything in d = anything, so that we can better exploit these stringy dualities. By the time all is said and done, we should have a solid understanding of the allowed particle content residing in extended theories of supersymmetry, as well as an idea of how the dimensionality of spacetime affects the super-Poincaré algebra.

We accomplish this in three easy steps: In section 2 we construct the most general superysmmetric algebra in four dimensions. We find that the familiar algebra of $\mathcal{N} = 1$ SUSY is modified slightly in the presence of additional spinor supercharges, in particular by the appearance of "central charges". Section 3 takes a careful look at how the super-Poincaré algebra is realized in a quantum field theory. We find massless and massive states that transform as "supermultiplets"—irreducible representations of the SUSY algebra. Generalizing features of the algebra, we find that many of these multiplets are in fact trivially related. In section 4 we change gears and explore how the dimensionality of spacetime influences our supersymmetric theory. We discover that the most effective way of classifying supersymmetric theories in *any* dimension is not by $\mathcal{N}$, but by the number of independent supercharges contained in the theory. This leads us to an investigation of spinors in various dimensions, with somewhat surprising results. Pooling our observations, we learn that we can efficiently move from a higher dimensional SUSY theory to a lower dimensional one through toroidal compactification, which is both practical and sort of fun. Finally, we conclude in section 5 with an executive summary and a few quick comments.

A Note on Conventions

Throughout this review, we opt for the "east coast" metric where $g_{\mu\nu} = \mathrm{diag}(-,+,+,+)$ in four dimensional Minkowski space. Weyl spinors will be manipulated with the aid of van der Waerden notation, where the two components of a left handed Weyl spinor are denoted ψ_α and those of a right handed Weyl spinor by $\bar{\psi}^{\dot{\alpha}}$. An introduction to this notation can be found in [3, 4].

2. The Super(er) Algebra

We obtain the super-Poincaré algebra in four dimensions by combining the familiar generators of the Poincaré group (P^μ and $M^{\mu\nu}$) with a set of spinorial supercharges Q^A_α and $\bar{Q}_{\dot{\alpha}A} = (Q^A_\alpha)^\dagger$. As indicated by the fermionic nature of the supercharges, the resulting algebra is $\mathbb{Z}_2$ graded, with the Q's odd and the P's and M's even under the grading (see figure 1). The most general supersymmetric algebra then follows from a few elementary considerations:

Figure 1: Pictorial representation of the $\mathbb{Z}_2$ grading of the super-Poincaré algebra. Generators even under the grading can be thought of as "positively charged" under the algebra, while those that are odd carry "negative" charge. This mnemonic is useful for categorizing commutators. Intuitively, $[+,+] = +$, $[+,-] = [-,+] = -$, and $\{-,-\} = +$.

The supercharges Q^A_α generate supersymmetric transformations that leave SUSY actions invariant, and are thus conserved in the usual Noether sense. Accordingly,

$$\left[H, Q^A_\alpha\right] = \left[P^0, Q^A_\alpha\right] = 0 \tag{2.1}$$

Lorentz covariance then requires

$$\left[P^\mu, Q^A_\alpha\right] = \left[P^\mu, \bar{Q}_{\dot{\alpha}A}\right] = 0 \tag{2.2}$$

Furthermore, since the supercharges are taken to be left and right Weyl spinors, it must be that their behavior under boosts and rotations is given by

$$\left[Q^A_\alpha, M^{\mu\nu}\right] = (\sigma^{\mu\nu})_\alpha{}^\beta Q^A_\beta \tag{2.3}$$

and

$$\left[\bar{Q}^{\dot{\alpha}}_A, M^{\mu\nu}\right] = (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \bar{Q}^{\dot{\beta}}_A \tag{2.4}$$

Where the $\sigma^{\mu\nu}$ are a two dimensional representation of the Lorentz generators which satisfy the same commutation relations as the $M^{\mu\nu}$:

$$\left[\sigma^{\mu\nu}, \sigma^{\gamma\delta}\right] = i\left(g^{\mu\gamma}\sigma^{\nu\delta} - g^{\nu\gamma}\sigma^{\mu\delta} - g^{\mu\delta}\sigma^{\nu\gamma} + g^{\nu\delta}\sigma^{\mu\gamma}\right) \tag{2.5}$$

In terms of the familiar Pauli matrices $\vec{\sigma}$, we can define the vectors $(\sigma^\mu)_{\alpha\dot{\beta}} = (\mathbb{1}, \vec{\sigma})_{\alpha\dot{\beta}}$ and $(\bar{\sigma}^\mu)^{\dot{\alpha}\beta} = (\mathbb{1}, -\vec{\sigma})^{\dot{\alpha}\beta}$. From these, an explicit representation of the $\sigma^{\mu\nu}$ is given by

$$\begin{aligned}
(\sigma^{\mu\nu})_\alpha{}^\beta &= \frac{i}{4}\left(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu\right)_\alpha{}^\beta \\
(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} &= -\frac{i}{4}\left(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu\right)^{\dot{\alpha}}{}_{\dot{\beta}}
\end{aligned} \tag{2.6}$$

We also need the odd-odd sector of the algebra, which is written in terms of anticommutators. Consider first the object $\{Q, \bar{Q}\}$, which is the symmetric combination of a left and right handed Weyl spinor. Since the Lorentz group is locally isomorphic to $SU(2) \otimes SU(2)$, we customarily label its representations by the ordered pair (s_+, s_-), which catalogues how it transforms under one $SU(2)$ or the other. In this language, a left Weyl spinor is a $(\frac{1}{2}, 0)$, a right Weyl spinor is a $(0, \frac{1}{2})$, and thus the anticommutator $\{Q, \bar{Q}\}$ transforms like $(\frac{1}{2}, \frac{1}{2})$—a vector. Since the algebra must close, we then know that this expression is proportional to the translation generator P^μ, and accordingly the only properly covariant combination is

$$\left\{Q^A_\alpha, \bar{Q}_{\dot{\beta}B}\right\} = -2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^A_B \tag{2.7}$$

The factor of two is conventional, and the identity matrix on the indices A, B is a consistent (but basis dependent) choice.

We also need to work out the commutator of two $Q's$, which is again simple group theory. The combination $\{Q, Q\}$ must transform like $(\frac{1}{2}, 0) \otimes (\frac{1}{2}, 0) = (0, 0)_A \oplus (1, 0)_S$ under the Lorentz group. Through copious use of the Jacobi identity, it is possible to show (e.g. [5]) that the symmetric (1,0) piece of the decomposition vanishes, leaving us with a term antisymmetric on the indices α, β:

$$\left\{Q^A_\alpha, Q^B_\beta\right\} = 2\epsilon_{\alpha\beta} Z^{AB} \tag{2.8}$$

Because the anticommutator is (by definition) symmetric under exchange of α, A and β, B, we discover that the Z must be antisymmetric with respect to A and B. Furthermore, we can again appeal to the Jacobi identity to show

$$\begin{aligned} 0 &= \left[\{Q^A_\alpha, Q^B_\beta\}, P^\mu\right] + \left\{[P^\mu, Q^A_\alpha], Q^B_\beta\right\} - \left\{[Q^B_\beta, P^\mu,], Q^A_\alpha\right\} \\ &= 2\epsilon_{\alpha\beta}\left[Z^{AB}, P^\mu\right] \\ &= \left[Z^{AB}, P^\mu\right] \end{aligned} \tag{2.9}$$

similarly, we find

$$\left[M^{\mu\nu}, Z^{AB}\right] = 0 \tag{2.10}$$

$$\left[Q, Z^{AB}\right] = 0 \tag{2.11}$$

$$\left[\bar{Q}, Z^{AB}\right] = 0 \tag{2.12}$$

Since Z commutes with every other generator in the algebra, it gets the special title *central charge*. It is important to note that in the (likely) more familiar framework of $\mathcal{N} = 1$ SUSY, antisymmetry forces the central charge to vanish. Thus, this is our first qualitatively new feature of an extended ($\mathcal{N} > 1$) supersymmetric algebra.

Of course our algebra will not close without the rest of the even-even sector, which is just the tried and true Poincaré algebra. For completeness, we have

$$[P^\mu, P^\nu] = 0 \tag{2.13}$$

$$[M^{\mu\nu}, P^\gamma] = i\left(g^{\gamma\mu}P^\nu - g^{\gamma\nu}P^\mu\right) \tag{2.14}$$

and (2.5) with $\sigma \to M$.

2.1 One and the Same

The super-Poincaré algebra can also enjoy the slightly less obvious R symmetries. These symmetries are related to the ability to rotate one supercharge into another, and are thus more technically categorized as *automorphisms* of the algebra. These automorphisms may generically consist of an abelian group $U(1)_R$ signifying the ability to rotate each supercharge by a global phase, and/or (in extended algebras) an $SU(\mathcal{N})_R$. The latter results from the invariance of the algebra under a scrambling of the $\mathcal{N}$ supercharges by an arbitrary unitary $\mathcal{N}\times\mathcal{N}$ matrix with unit determinant.

The appearance of these additional symmetries in principle allows us the ability to assign a definite R-charge to fields transforming under irreducible representations of the algebra. Importantly, because the generators of these automorphisms R commute with all generators *except* the supercharges, it happens that different fields *within* a supermultiplet can carry different R-charge. Classically this can occur depending on the dynamics of (e.g) the superpotential, however anomalies in the full quantum theory often eradicate or further constrain these automorphisms [6].

3. Placing Particles

With the technical details behind us, we are at last well poised to do some physics. Our immediate goal will be to catalogue the (on-shell) particle content for various SUSY theories in four dimensions. Once the supersymmetry is imposed, it is obvious that these particles must appear in multiplets of the SUSY algebra. So really, our task is simply to enumerate all possible irreducible representations for a supersymmetric theory with given $\mathcal{N}$.

As a matter of convenience, we will accomplish this by considering massless and massive representations independently. The gist of the program will be to pick a convenient Lorentz frame, and rephrase the SUSY algebra in terms of a Clifford algebra with run-of-the-mill creation and annihilation operators. The allowed states in a given representation are then indexed via the standard "highest weight" method, where one essentially descends the entire multiplet with lowering operators until the lowest state in the multiplet (which is annihilated by any further lowering) is obtained.

3.1 Skinny

If the preceding paragraph sounds somewhat cryptic, the simple example of the massless representations will certainly make it more concrete. A convenient Lorentz frame for these states is the null frame where

$$\{P^\mu\} = (E, 0, 0, E) \tag{3.1}$$

so that (2.7) becomes

$$\left\{Q^A_\alpha, \bar{Q}_{\dot\beta B}\right\} = -2\left(-P^0\cdot\mathbb{1} + P^3\cdot\sigma_3\right)_{\alpha\dot\beta}\delta^A_B = \delta^A_B\begin{pmatrix}0 & 0\\ 0 & 4E\end{pmatrix}_{\alpha\dot\beta} \tag{3.2}$$

Equation (3.2) contains quite a bit of useful information. Immediately, we see that when $A = B$ and $\alpha = \dot\beta = 1$

$$\{Q^A_1, \bar{Q}_{1A}\} = 0 \tag{3.3}$$

which implies

$$\begin{aligned} 0 &= \langle\gamma|\left\{Q^A_1, \bar{Q}_{1A}\right\}|\gamma\rangle \\ &= \langle\gamma|Q^A_1\bar{Q}_{1A}|\gamma\rangle + \langle\gamma|\bar{Q}_{1A}Q^A_1|\gamma\rangle \\ &= ||\bar{Q}_{1A}|\gamma\rangle||^2 + ||Q^A_1|\gamma\rangle||^2 \end{aligned} \tag{3.4}$$

On physical grounds, we are interested in the case where our particles live in unitary representations. This requires that the states $|\gamma\rangle$ are positive definite, which (along with (3.4)) forces $Q_1^A = \bar{Q}_{1A} = 0$ to hold as operator equations. We can then insert this observation into (2.8) to show that

$$\langle\gamma| \left\{Q_1^A, Q_2^B\right\} |\gamma\rangle = 2\langle\gamma|Z^{AB}|\gamma\rangle = 0 \tag{3.5}$$

which similarly implies that $Z^{AB} = 0$ holds as an operator equation in the massless sector of our theory. Accordingly, we discover that massless states can not carry central charge, a fact that will greatly simplify our present undertaking.

Summarizing the above, we have found that the only interesting supercharges acting on massless states are the Q_2^A and $\bar{Q}_{2B}$. It will therefore be prudent to examine them a bit closer. If we define

$$\hat{a}_A \equiv \frac{1}{2\sqrt{E}} Q_2^A \tag{3.6}$$

$$\hat{a}_A^\dagger \equiv \frac{1}{2\sqrt{E}} \bar{Q}_{2A} \tag{3.7}$$

it is easy to see that the relevant part of equations (3.2) and (2.8) become

$$\left\{\hat{a}_A, \hat{a}_B^\dagger\right\} = \delta_{AB} \tag{3.8}$$

$$\left\{\hat{a}_A, \hat{a}_B\right\} = 0 \tag{3.9}$$

$$\left\{\hat{a}_A^\dagger, \hat{a}_B^\dagger\right\} = 0 \tag{3.10}$$

$$\tag{3.11}$$

which is just the familiar statement that $\hat{a}$ and $\hat{a}^\dagger$ are legitimate raising and lowering operators of a Clifford algebra. Furthermore, since the helicity is measured by the time component of the Pauli-Lubanski vector $W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma}$ (modulo the norm of the state's 3-momentum), in our frame the helicity operator is just $\Sigma = M^{12}$. Revisiting (2.3), we then find

$$[\Sigma, \hat{a}_A] = \frac{1}{2\sqrt{E}}\left[M^{12}, Q_2^A\right] = \frac{1}{2\sqrt{E}}(\sigma^{12})_2{}^2 Q_2^A = -\frac{1}{2}\hat{a_A} \tag{3.12}$$

which is merely the statement that the operator $\hat{a}_A$ lowers the helicity of a state by $\frac{1}{2}$. Explicitly, consider a state of definite helicity λ, so $\Sigma|\lambda\rangle = \lambda|\lambda\rangle$. If we act on this state with $\hat{a}_A$ we get a new state $|\Lambda\rangle \equiv \hat{a}_A|\lambda\rangle$ with helicity

$$\Sigma|\Lambda\rangle = \Sigma\hat{a}_A|\lambda\rangle = \hat{a}_A\Sigma|\lambda\rangle + [\Sigma, \hat{a}_A]|\lambda\rangle = (\lambda - \frac{1}{2})|\Lambda\rangle \tag{3.13}$$

Not surprisingly, nearly identical calculations show that the remaining set of operators, $\hat{a}_A^\dagger$, raise the helicity of a state by $\frac{1}{2}$.

We now have all the ingredients necessary to cook up some irreducible representations. The recipe is simple: choose a "highest weight" state $|\lambda\rangle$ with definite helicity λ. The state is "highest" in the sense that $\hat{a}_A^\dagger|\lambda\rangle = 0$ by construction. We now descend the multiplet in its entirety by acting on $|\lambda\rangle$ with all possible non-vanishing combinations of the $\hat{a}_A$. Generically, then, the states of this massless representation have the form

$$\left|\lambda - \frac{M}{2}, [A, B, \ldots, M]\right\rangle \sim \hat{a}_A \hat{a}_B \ldots \hat{a}_M |\lambda\rangle \tag{3.14}$$

As indicated, these states are anti-symmetric under exchange of the "extended" indices $A, B, \ldots, M$ (by virtue of the Clifford algebra), which means each of the $\mathcal{N}$ lowering operators can appear at most once. For a state with M distinct $\hat{a}$'s, there is thus a degeneracy of $\binom{\mathcal{N}}{M}$ since the overall phase of the state doesn't matter. Furthermore, the multiplet will obviously contain all helicities from λ to $\lambda - \frac{\mathcal{N}}{2}$, for a grand total of

$$\sum_{M=0}^{\mathcal{N}} \binom{\mathcal{N}}{M} \equiv \sum_{M=0}^{\mathcal{N}} \frac{\mathcal{N}!}{M!(\mathcal{N}-M)!} = 2^{\mathcal{N}} \tag{3.15}$$

states.

Since all theories of physical interest happen to be CPT invariant, we will always require the particle content in each massless supermultiplet of our theory to be symmetric under $\lambda \to -\lambda$. Because the states described by (3.14) only satisfy this requirement when $\lambda = \mathcal{N}/4$, it is typically necessary to supplement a given massless multiplet with the direct sum of its CPT conjugate multiplet with helicities $-\lambda$ to $-\lambda + \frac{\mathcal{N}}{2}$. It is worth noting that the CPT self-conjugate theory with $\mathcal{N} = 4$ is the "maximally extended" Yang-Mills theory in four dimensions, as it contains the largest number of supercharges consistent with a theory describing massless spin 1 particles ($\lambda - \mathcal{N}/2 = -1$ for $\lambda = 1$). Analogously, we find that CPT self-conjugate $\mathcal{N} = 8$ theory is the maximally extended supergravity theory in four dimensions, since massless particles with spin > 2 necessarily appear for $\mathcal{N} > 8$. We will revisit these multiplets and more in gory detail in section 3.4.

3.2 Fat

We next turn to massive particle content of supersymmetric theories. In general, the massive sector carries central charge, so it is not immediately obvious that we can rephrase the SUSY algebra in terms of creation and annihilation operators as we did for the massless representations. Through clever change of basis, we can mitigate this complication, but the resulting mess of operators and indices can take some time to parse. The essential result of our impending calculation is that we can indeed arrive at a Clifford algebra in the presence of a central charge, but *which* algebra we obtain is sensitive to the particle's mass and charge.

We begin as in section 3.1, by choosing a convenient Lorentz frame for our analysis. The obvious choice is the massive particle's rest frame, where

$$\{P^\mu\} = (M, 0, 0, 0) \tag{3.16}$$

in which case (2.7) becomes

$$\left\{Q^A_\alpha, \bar{Q}_{\dot{\beta}B}\right\} = 2M\delta^A_B\,\delta_{\alpha\dot{\beta}} \tag{3.17}$$

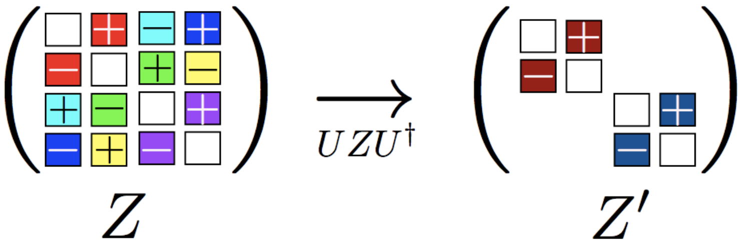

Figure 2: Cartoon of the "diagonalization" of an antisymmetric matrix. The central charges Z are brought to a block diagonal form Z' where each block is $i\sigma_2$ times a new charge ($Z_{\tilde{a}}$ in the text). Different charges correspond to different colors in the $\mathcal{N} = 4$ decomposition shown above.

Importantly, (3.17) is invariant under an arbitrary $SU(\mathcal{N})_R$ rotation of the supercharges, where $Q^A_\alpha \to U_{AB} Q^B_\alpha$ and $\bar{Q}_{\dot{\beta}A} \to \bar{Q}_{\dot{\beta}B} U^{\dagger BA}$, and $U^\dagger U = \mathbb{1}$. We can exploit this invariance to "diagonalize" the central charges and bring (2.8) into a more congenial form. Of course the anti-symmetry of the Z^{AB} implies the closest we can get to a non-trivial (i.e. non-vanishing) diagonal matrix is something of the form $Z_{\tilde{a}}(i\sigma_2 \otimes \mathbb{1})$. The $Z_{\tilde{a}}$ are real positive numbers, $\mathbb{1}$ is the $\mathcal{N} \times \mathcal{N}$ identity matrix, and the $i\sigma_2$ is necessary to preserve the anti-symmetry of the Z^{AB} which is unchanged by special unitary transformations. In fact, it is not hard to see that we can only bring the central charges to this form for even values of $\mathcal{N}$—when $\mathcal{N}$ is odd, we simply append an extra row and column of zeros. This decomposition is illustrated in figure 2.

To actually manipulate this matrix, it is convenient to divvy up the capitol indices like $A = \{a, \tilde{a}\}$ where $a = 1, 2$ and $\tilde{a} = 1, 2, \ldots, \mathcal{N}/2$. Note that we are specializing to the case where $\mathcal{N}$ is even; the extension to general $\mathcal{N}$ is trivial but notationally cumbersome. In this decomposition, our central charges are expressed like

$$Z^{AB} \equiv Z^{\{a,\tilde{a}\}\{b,\tilde{b}\}} = Z_{\tilde{b}}\, \epsilon^{ab}\, \delta^{\tilde{a}\tilde{b}} \tag{3.18}$$

where it is important to note that the repeated index $\tilde{b}$ is *not* summed. Taking stock of what we have accomplished, the relevant pieces of the algebra become

$$\left\{ Q^A_\alpha, \bar{Q}_{\dot{\beta}B} \right\} = 2M \delta^a_b\, \delta^{\tilde{a}}_{\tilde{b}}\, \delta_{\alpha\dot{\beta}} \tag{3.19}$$

$$\left\{ Q^A_\alpha, Q^B_\beta \right\} = 2Z_{\tilde{b}} \epsilon_{\alpha\beta}\, \epsilon^{ab}\, \delta^{\tilde{a}\tilde{b}} \tag{3.20}$$

So really, it just looks like we have made everything much more complicated. In fact, if we make the inspired operator redefinitions (momentarily bending a few spinor index rules)

$$\hat{a}^{\tilde{a}}_\alpha = \frac{1}{2}\left(Q^{1\tilde{a}}_\alpha + \epsilon_{\alpha\beta} \bar{Q}_{\dot{\gamma}2\tilde{a}} \delta^{\dot{\gamma}\beta} \right) \tag{3.21}$$

$$\hat{b}^{\tilde{a}}_\alpha = \frac{1}{2}\left(Q^{1\tilde{a}}_\alpha - \epsilon_{\alpha\beta} \bar{Q}_{\dot{\gamma}2\tilde{a}} \delta^{\dot{\gamma}\beta} \right) \tag{3.22}$$

we arrive at a great success. The algebra of these improved operators is straightforward to work out. For example

$$\begin{aligned}
\left\{\hat{a}^{\tilde{a}}_{\alpha}, (\hat{a}^{\tilde{b}}_{\beta})^{\dagger}\right\} &= \frac{1}{4}\left\{Q^{1\tilde{a}}_{\alpha} + \epsilon_{\alpha\rho}\bar{Q}_{\dot{\gamma}2\tilde{a}}\delta^{\dot{\gamma}\rho}, \bar{Q}_{\dot{\beta}1\tilde{b}} + \epsilon_{\dot{\beta}\dot{\sigma}}Q^{2\tilde{b}}_{\lambda}\delta^{\dot{\sigma}\lambda}\right\} \\
&= \frac{1}{2}\left\{Q^{1\tilde{a}}_{\alpha}, \bar{Q}_{\dot{\beta}1\tilde{b}}\right\} + \frac{1}{2}\left\{Q^{1\tilde{a}}_{\alpha}, Q^{2\tilde{b}}_{\lambda}\right\}\epsilon_{\dot{\beta}\dot{\sigma}}\,\delta^{\dot{\sigma}\lambda} \\
&= (M + Z_{\tilde{a}})\;\delta^{\tilde{a}}_{\tilde{b}}\,\delta_{\alpha\dot{\beta}} \qquad (3.23)
\end{aligned}$$

and the rest are just as easy. Up to Hermitian conjugation,

$$\left\{\hat{b}^{\tilde{a}}_{\alpha}, (\hat{b}^{\tilde{b}}_{\beta})^{\dagger}\right\} = (M - Z_{\tilde{a}})\;\delta^{\tilde{a}}_{\tilde{b}}\,\delta_{\alpha\dot{\beta}} \qquad (3.24)$$

$$\left\{\hat{a}, \hat{b}^{\dagger}\right\} = 0 \qquad (3.25)$$

$$\left\{\hat{a}, \hat{b}\right\} = 0 \qquad (3.26)$$

$$\left\{\hat{a}, \hat{a}\right\} = 0 \qquad (3.27)$$

$$\left\{\hat{b}, \hat{b}\right\} = 0 \qquad (3.28)$$

Lo and behold, we have arrived at yet another Clifford algebra—we are back on familiar ground. Before we revisit the tired task of enumerating the states in a given representation, let's pause for a second to reflect on the importance of (3.24). Sandwiching this expression between a unitary particle state $|\psi\rangle$, and choosing $\tilde{a} = \tilde{b}$, $\alpha = \dot{\beta}$ we find

$$\begin{aligned}
\langle\psi|\left\{b, b^{\dagger}\right\}|\psi\rangle &= ||b^{\dagger}|\psi\rangle||^2 + ||b|\psi\rangle||^2 \\
&= M - Z \qquad (3.29)
\end{aligned}$$

Since the left hand side of (3.29) is clearly positive, we are led to an example of the ubiquitous Bogomolnyi-Prasad-Sommerfield (BPS) bound, which constrains the allowed charge based on the mass:

$$M \geq Z_{\tilde{a}} \qquad (3.30)$$

we will study the special states that saturate this bound in section 3.3.

For now, consider only those representations whose central charge is strictly less than their mass. In this familiar case, we have a Clifford algebra generated by the $2\mathcal{N}$ creation operators ($(\hat{a}^{\tilde{a}}_{\alpha})^{\dagger}, (\hat{b}^{\tilde{a}}_{\alpha})^{\dagger}$ for $\alpha = 1, 2$ and $\tilde{a} = 1, 2, \ldots, \mathcal{N}/2$) and their associated annihilation operators. Incidentally, it is not hard to see that this is identical to the algebra we would have obtained in the absence of any central charges, so in this respect we have taken a rather extravagant detour. Nonetheless, we are now free to construct our irreducible representations by following the same prescription employed in section 3.1. We start with a highest weight state of definite spin $|\Omega\rangle$ and apply the lowering operators in all possible ways until the state is annihilated. In this massive sector, however, matters are slightly complicated by the fact that the $\hat{a}$'s and $\hat{b}$'s transform like $(\frac{1}{2}, 0)$ under the Lorentz group, and hence the spin of a given state in the representation with M lowering operators acting on $|\Omega\rangle$ is no longer simply $M/2$ less than the spin of $|\Omega\rangle$. Annoyingly, it turns out that the number of states of a particular spin in a massive supermultiplet is given by the dimension of an antisymmetric representation of the unitary symplectic group of rank $\mathcal{N}$, $USp(2\mathcal{N})$. For details on this non-obvious technicality, see [7]. Of course for any given spin j, there is an additional $(2j+1)$-fold degeneracy familiar from the study of spin in quantum mechanics. Again, we will catalogue some massive multiplets of interest in section 3.4.

3.3 Short

The only remaining multiplets in our theory arise for the special values of central charge that saturate (3.30). These aptly named BPS states actually have a very simple structure. Pretend that $Z_{\tilde{a}} = M$ for $\tilde{a} = 1, 2, \ldots, p$. For these values of $\tilde{a}$ the anti-commutator $\{\hat{b}^{\tilde{a}}, (\hat{b}^{\tilde{a}})^\dagger\}$ must vanish as per (3.24), which immediately shows $\hat{b}^{\tilde{a}} = (\hat{b}^{\tilde{a}})^\dagger = 0$ as operators. The reasoning is identical to (3.4). The upshot is that our algebra is now generated by the $\mathcal{N}/2$ $\hat{a}$'s, the $\mathcal{N}/2 - p$ $\hat{b}$'s that don't saturate the bound, and their conjugates. Thus we find that the BPS states belong to representations generated by $2\mathcal{N} - 2p$ creation operators (remember the $\hat{a}^\dagger$ and $\hat{b}^\dagger$ also carry the index $\alpha = 1, 2$), and these representations clearly contain less states then the case where $Z_{\tilde{a}}$ is strictly less than the mass. For this reason, we say that BPS saturation leads to multiplet shortening, and we sometimes use the adjectives "long" and "short" to distinguish massive mutiplets with $Z_{\tilde{a}} < M$ and $Z_{\tilde{a}} = M$ respectively. In the following section we will see that we can often relate the short multiplets of an extended SUSY theory to massless multiplets in the same theory, via the Higgs mechanism.

3.4 Examples

At long last it is time to reap the benefits of our hard work and explicitly display a few SUSY multiplets. The massless sector is particularly interesting, as it is here that we discover the content of supersymmetric gauge theories. Figure 3 illustrates the number of states for each helicity for a few popular theories. Translating the table into familiar field

λ	$\mathcal{N}=1$ Vector	$\mathcal{N}=1$ Chiral	$\mathcal{N}=2$ Vector	$\mathcal{N}=2$ Hyper	$\mathcal{N}=4$ Vector
1	1	0	1	0	1
1/2	1	1	2	2	4
0	0	$1+1$	$1+1$	4	6
$-1/2$	1	1	2	2	4
-1	1	0	1	0	1
# of States	$2+2$	$2+2$	$4+4$	8	16

Figure 3: Some common massless multiplets residing in supersymmetric gauge theories. The table stops at $\mathcal{N} = 4$, as this is the maximally extended Yang-Mills theory in four dimensions. The "+" symbols serve as a reminder that in most cases we must combine a mulitplet with its CPT conjugate to obtain a CPT invariant theory. For CPT self-dual hypermultiplets, like the one that appears in $\mathcal{N} = 2$, a "1/2" hypermultiplet is in fact allowed, consistent with our binomial distribution counting rules established previously.

content, we thus learn that (for example) an $\mathcal{N} = 2$ hypermultiplet contains two left Weyl fermions, λ^i_α and two complex scalars $\chi_\pm$, while an $\mathcal{N} = 4$ vector multiplet consists of a vector A_μ, four left Weyl fermions λ^i_α, and 6 real scalars φ^a.

Restricting our attention to massive representations with spin $s \leq 1$, the allowed *long* multiplets in four dimensional superymmetry are tabulated in figure 4. It is interesting

S	$\mathcal{N}=1$ Vector	$\mathcal{N}=1$ Chiral	$\mathcal{N}=2$ Vector
1	1	0	1
1/2	2	1	4
0	1	2	5
# of States	8	4	16

Figure 4: A few long massive multiplets containing states with spins less than 1. Note that for each spin there is a further (2s+1) fold degeneracy, which is reflected in the total number of states shown above.

to compare these representations with their BPS counterparts. For $\mathcal{N} = 1$ there can be no central charge, and hence no such thing as a short multiplet. For $\mathcal{N} = 2$, there is one central charge, and for $\mathcal{N} = 4$ there are two (see figure 2). Let us restrict our attention to the interesting case where all central charges saturate the BPS bound, effectively halving the number of creation operators in our theory. The short multiplets of interest are then found in figure 5.

BPS

S	$\mathcal{N}=2$ Vector	$\mathcal{N}=2$ Hyper	$\mathcal{N}=4$ Vector
1	1	0	1
1/2	2	2	4
0	1	4	5
# of States	8	8	16

Figure 5: Important BPS saturated multiplets in supersymmetry. Commonalities relate these representations to others in the theory.

Importantly, by studying figures 3, 4, and 5 we can begin to see how all the lines of algebra in the preceding three sections fit together. First, note that the spectra of the $\mathcal{N} = 4(2)$ BPS vector multiplet coincides with the $\mathcal{N} = 2(1)$ massive vector multiplet. This is as expected, since BPS saturation basically removes half the available supercharges, and we thus end up studying representations of $\mathcal{N}/2 = 2(1)$ supersymmetry. Moreover, consider the Higgs mechanism in, say, $\mathcal{N} = 4$. We begin with the purple multiplet in figure 3, and allow a scalar to condense. As symmetry is spontaneously broken, the vector boson will consume a scalar so that it carries the number of degrees of freedom required of a massive spin one particle. It is not surprising, then, that there exists a massive SUSY multiplet in the $\mathcal{N} = 4$ theory with one vector, 4 fermions, and only *five* scalars. This is just the purple BPS vector multiplet of figure 5! It is simple to verify that this method of counting reproduces the BPS vector multiplet in the $\mathcal{N} = 2$ theory as well, highlighting its utility as a mnemonic. Evidently, strategic memorization of a few key multiplets coupled with a basic understanding of the algebra can get us pretty far.

4. When Four isn't Enough

In order to get the most bang for our SUSY buck, we would also like to get a handle on what it means to have a supersymmetric theory in *any* number of spacetime dimensions. This is true for a number of reasons, the most compelling of which all seem to revolve around realizations of supersymmetry in string theory. Popular examples include supersymmetry on the two-dimensional string world sheet, and gauge/gravity duals that relate supergravity in ten dimensions to globally supersymmetric field theories in four.

Unfortunately, generalizing the super-Poincaré algebra to arbitrary dimensions is not as easy as readjusting the range of our generator's Lorentz indices. Certainly we can do this without issue for the Poincaré group. The real issue, of course, is how we are going to deal with the supercharges. As a consequence of their spinorial nature, the properties of these supercharges are highly sensitive to the dimension of the spacetime they reside in.

The fact that the dimension of space strongly affects the properties of the objects generating our algebra suggests that the defining characteristic of our SUSY theories should no longer be the value of $\mathcal{N}$, but rather the number of independent SUSY charges that generate the algebra. Let's be concrete: for $\mathcal{N} = 1$ in four dimensions, there are 4 independent supersymmetry generators: $Q_1, Q_2, \bar{Q}^{\dot{1}}$ and $\bar{Q}^{\dot{2}}$. The observation that we can fit these four generators into $\mathcal{N} = 1$ complex Weyl fermion is irrelevant to the construction of SUSY multiplets. What matters is how many batches of creation and annihilation operators we have at the end of the day, a fact we became intimately familiar with in the preceding sections. From a cataloguing standpoint, it is thus clear that it would be useful to understand the minimum number of supersymmetry generators we can have in a given dimension. This in turn is trivially related to the minimum spinor dimension in a given dimension, which we now attempt to work out.

4.1 Spinning in d Dimensions

Our first task is to take a closer look at the representations of $SO(d-1, 1)$, which is to say the dimensions of the Dirac spinors transforming under the Clifford algebra

$$\{\Gamma^\mu, \Gamma^\nu\} = 2g^{\mu\nu} \tag{4.1}$$

Fortunately for us, we have more or less spent the last ten pages wading through identical algebras, so this isn't actually so daunting. Let's see if we can hit the highlights without being excruciatingly redundant...

As we have seen, dissecting a Clifford algebra really amounts to constructing raising and lowering operators and using them to walk through representations á la "highest weights". To make life simpler, lets focus on spacetimes of even dimension $d = 2k+2$ (in this section we follow the notation of [8], where further details—including d odd—are abundant). With a glance at (4.1) it is obvious that we can divvy up the Γ^μ into genuine raising and lowering operators by defining

$$\Gamma^{0\pm} = \frac{1}{2}\left(\pm\Gamma^0 + \Gamma^1\right) \tag{4.2}$$

$$\Gamma^{a\pm} = \frac{1}{2}\left(\Gamma^{2a} \pm i\Gamma^{2a+1}\right) \tag{4.3}$$

where $a = 1, 2, \ldots, k$. This is obviously the right choice, since it renders (4.1) into the form

$$\left\{\Gamma^{i+}, \Gamma^{j-}\right\} = \frac{1}{4}\left[\left\{\Gamma^{2i}, \Gamma^{2j}\right\} + \left\{\Gamma^{2i+1}, \Gamma^{2j+1}\right\}\right] = \delta^{ij} \tag{4.4}$$

$$\left\{\Gamma^{i+}, \Gamma^{j+}\right\} = 0 \tag{4.5}$$

$$\left\{\Gamma^{i-}, \Gamma^{j-}\right\} = 0 \tag{4.6}$$

Comparing to (3.8) we remember that we have already "been there" and "done that". We construct our representation by acting on a highest weight spinor state in all possible ways with the $2k+1$ lowering operators, Γ^{i-}, until the state is annihilated. In this way, we obtain a representation of dimension 2^{k+1}, which we call the Dirac representation of the Lorentz group in $2k+2$ dimensions, $SO(2k+1, 1)$. To see that this is indeed a spinor representation (i.e. a representation with half integer spin) it is necessary to supplement this discussion with the analogue of (3.12). The construction of this spin operator is straightforward but not particularly exciting, and it will henceforth suffice to say that it gives the expected result.

For better or worse, this Dirac representation is in general reducible, and thus our work is not yet finished. Of course anyone who has ever used a projection operator to write a two component Weyl spinor in terms of a four component Dirac spinor in four dimensions (remember $\psi_+ = \frac{1}{2}(1+\gamma_5)\Psi$?) is already well acquainted with this fact. Generalizing this familiar expression to arbitrary even dimensions, we define a new chirality operator

$$\gamma_5 \to \tilde{\Gamma} \equiv i^{-k}\Gamma^0\Gamma^1 \ldots \Gamma^{2k+1} \tag{4.7}$$

Happily, this operator commutes with spin, and it is easy to see that $\tilde{\Gamma}^2 = \mathbb{1}$. Accordingly, the eigenvalues of $\tilde{\Gamma}$ are ± 1, and we can further subdivide states of definite spin into states of definite chirality. In other words, in even dimensions we can take our 2^{k+1} states and divide them into a 2^k dimensional *left* Weyl representation and a 2^k dimensional *right* Weyl representation. It is worth noting that in odd dimensions no such decomposition takes place because, roughly speaking, $\tilde{\Gamma}$ is the identity—there is no non-trivial chirality operator.

In addition to the Weyl criteria, there is also a Majorana condition which is more or less a reality constraint. This suggests it will be worth our time to figure out how our algebra behaves under complex conjugation. In four dimensions, we typically take γ^2 to be the only imaginary generator, and thus by virtue of the Clifford algebra we can use it to conjugate the γ^μ, because $\gamma^2\gamma^\mu(\gamma^2)^{-1} = \gamma^2\gamma^\mu\gamma^2 = -(\gamma^\mu)^*$. In $d = 2k+2$ dimensions, we can use the same idea with only a few minor alterations. First, because of our definitions in (4.2) and the fact that we have chosen a basis where the raising and lowering operators $\Gamma^{a\pm}$ are real, we see that the odd Γ^μ (excluding Γ^1) must be imaginary. By analogy to the familiar four dimensional case, we can then define conjugation operators, like

$$C_1 = \Gamma^3\Gamma^5\ldots\Gamma^{2k+1} \tag{4.8}$$
$$C_2 = \tilde{\Gamma}C_1 \tag{4.9}$$

which have the effect $C_1\Gamma^\mu C_1^{-1} = (-1)^k\,\Gamma^{\mu *}$ and $C_2\Gamma^\mu C_2^{-1} = (-1)^{k+1}\,\Gamma^{\mu *}$ as a result of (4.1). We can use these operators to relate a spinor to its complex conjugate, since ψ and $C_i^{-1}\psi^*$ have the same Lorentz properties. This follows trivially from the observation that if $[M^{\mu\nu},\psi] = S^{\mu\nu}\psi$ where $S^{\mu\nu} = -\frac{i}{4}[\Gamma^\mu,\Gamma^\nu]$, then

$$C_i S^{\mu\nu} C_i^{-1} = -\frac{i}{4}C_i\,[\Gamma^\mu,\Gamma^\nu]\,C_i^{-1} = -\frac{i}{4}\,[\Gamma^{\mu *},\Gamma^{\nu *}] = -(S^{\mu\nu})^* \tag{4.10}$$

The point of all this is that the "Majorana condition" $\psi^* = C_i\psi$ is Lorentz compatible, and so we can use it to further classify our Dirac spinors. It is easy to see that the Majorana condition implies $\psi = C_i^*\psi^* = C_i^*C_i\psi$, so this constraint is only consistent for operators that satisfy $C^*C = \mathbb{1}$. By explicit multiplication, we find

$$\begin{aligned} C_1^*C_1 &= \Gamma^{3*}\Gamma^{5*}\ldots\Gamma^{2k+1*}C_1 \\ &= (-1)^{k^2}C_1\Gamma^3\Gamma^5\ldots\Gamma^{2k+1}C_1^{-1}C_1 \\ &= (-1)^{k^2}(C_1)^2 \qquad (4.11) \\ &= (-1)^{k(k+1)/2} \qquad (4.12) \end{aligned}$$

and similar manipulations give $C_2^*C_2 = (-1)^{k(k-1)/2}$. Now finding the allowed $d = 2k+2$ consistent with a Majorana condition is a simple exercise in arithmetic. It turns out that we can use C_1 to constrain a Majorana spinor in $k = (0 \text{ or } 3) + 4n$ for integer n. Similarly, C_2 gives us a Majorana spinor in $k = (0 \text{ or } 1) + 4n$. Once odd dimensions are accounted for, it turns out that a Majorana condition is consistent in $d = 0, 1, 2, 3$ and 4, modulo 8.

The only remaining possibility is that the spinor of interest is somehow both a Weyl fermion *and* a Majorana fermion—a "Majorana-Weyl" fermion (or is it "Weyorana"?). For this to be the case, we obviously have to be in a dimension that allows both conditions simultaneously ($d = 2, 4, 8, 10$), but this is not enough. Since the chirality operator $\tilde{\Gamma}$ changes sign under conjugation by the C_i for k odd, the C_i can bring us from a left Weyl fermion to a right Weyl fermion in $d = 0$ modulo 4. Intuitively, this should not be allowed for a spinor that also has Majorana characteristics, as this object still needs to be "real" in the appropriate sense. Therefore, we learn that Majorana-Weyl fermions exist in $d = 2$ modulo 8.

This motley multitude of spinor results is organized in figure 6. Also included is the smallest number of real components a spinor carries in a given dimension, which we denote $\dim\psi_0$. This value is just twice the dimension of the Dirac representation in d dimensions (D_Γ), divided by 2 for a Weyl condition (2_W) and/or 2 for a Majorana constraint (2_M):

$$\dim\psi_0 = 2\frac{D_\Gamma}{2_W\cdot 2_M} \tag{4.13}$$

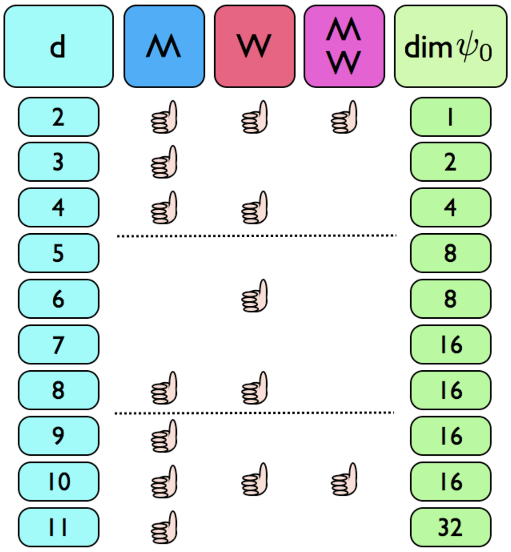

Figure 6: Spinors in different dimensions (d) may satisfy a Majorana condition (M), a Weyl condition (W), a Majorana-Weyl condition (MW), or neither. The minimum spinor dimension for a given d is the smallest number of real components that spinor can have, and is denoted dimψ_0. It is twice the size of the Dirac representation, divided by two for a Weyl condition or two for a Majorana condition.

Now that we know everything there is to know about the *fewest* number of supercharges we can fit into a spinor, it is interesting to wonder what is the *most*. This bound is necessarily phenomenological—otherwise we could just pack as many ψ_0's into our theory as our heart desired. Instead we are guided by the fact that the theories we care about contain fields with spin no larger than 2. As we saw in section 3.1, this observation allowed us to dub the four dimensional $\mathcal{N} = 8$ theory *maximally* extended supergravity. From figure 6, we note that the minimal spinor in four dimensions has 4 components, which gives us a theory of 32 supercharges. This bound is dimension independent—the spacetime dimension just tells us how the charges should be organized with respect to the Lorentz algebra. Furthermore, because we expect to be able to arrive at our four dimensional theory by compactifying higher dimensional theories over (for example) a torus, it is easy to see that there are no phenomenologically interesting SUSY theories in $d > 11$. This is evident from (4.13), which tells us that the smallest spinorial SUSY generator in 12 dimensions would necessarily contain $2 \times 2^6/2 = 64$ supercharges, overshooting by a factor of two.

4.2 Reductions

Just as the Higgs mechanism provided a handy mnemonic for relating the SUSY content of massless theories to the supermultiplets of a massive theory in section 3.4, we can use dimensional reduction to relate supersymmetric theories in different dimensions. The particulars of supersymmetric compactification are both fascinating and technical, but the overarching philosophy is fairly straightforward. Conceptually, we can imagine starting with a theory in d dimensional Minkowski space, and wrapping up $d-4$ of the dimensions into circles. This leaves us with a space that looks like $\mathcal{M}_d \to \mathcal{M}_4 \times S_1 \times S_1 \ldots \times S_1 = \mathcal{M}_4 \times T^{d-4}$, which is why this process is referred to as toroidal compactification. As is likely familiar, we are left with a four dimensional theory and a bevy of so-called "Kaluza-Klein modes". The question is, *which* theory are we left with?

Since the fields in our original d-dimensional theory transformed under representations of $SO(d-1,1)$, we can learn their fate in the four dimensional theory by working out how these representations descend to $SO(3,1)$. This is ideologically identical to working out how the representations of, say, $SU(5)$ decompose into representations of $SU(3) \times SU(2) \times U(1)$ like in grand unification, but slightly more involved. Without drowning in details, it is still wonderfully simple to see how this all transpires.

Consider a supersymmetric theory with 16 supercharges in ten dimensions. From figure 6, we see that we can fit these supercharges into one Majorana-Weyl fermion, so we are not-so-secretly studying $\mathcal{N} = 1$ SUSY in $d = 10$. In addition, let's make it a super Yang-Mills theory and focus on a vector supermultiplet, with one gauge field A_μ for $\mu = 0, 1, \ldots, 9$ and one Majorana-Weyl fermion λ_a for $a = 1, 2, \ldots, 16$. From the four dimensional perspective, the 16 supercharges persist, but they now reside in 4 (complex) Weyl fermions. This suggests that we have arrived at the $\mathcal{N} = 4$ theory in 4 dimensions. Furthermore, we can divvy up the ten dimensional vector into one four dimensional vector $\tilde{A}_\mu$ and 6 real scalars ϕ_i, like so:

$$\{A_\mu\} = \begin{pmatrix} \tilde{A}_0 \\ \tilde{A}_1 \\ \tilde{A}_2 \\ \tilde{A}_3 \\ \hline \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \end{pmatrix} \tag{4.14}$$

The suggestive form of (4.14) indicates that we have used the math fact $SO(9,1) \approx SO(3,1) \times SO(6)$ to decompose a vector representation of the Lorentz group in ten dimensions into representations of the four dimensional theory with a Lorentz $SO(3,1)$ label and an internal $SO(6)$ R-symmetry label. Schematically, we can display these representations like $(2s+1, 2s'+1)_R$, so we have found

$$A_\mu \to (\frac{1}{2}, \frac{1}{2})_1 + (0,0)_6 \tag{4.15}$$

under compactification. Of course the ten dimensional Majorana-Weyl fermions λ decompose as well, just like the ten dimensional spinor supercharge:

$$\lambda \to (\frac{1}{2}, 0)_4 \tag{4.16}$$

Taking stock of what we have learned, we see that the dimensional reduction of the gauge multiplet in ten dimensional $\mathcal{N} = 1$ theory gives us *precisely* the desired content of the gauge multiplet in the four dimensional $\mathcal{N} = 4$ theory (see figure 3). Moreover, since $SO(6)$ is homomorphic to $SU(4)$, it is clear that we have even reproduced the proper transformation properties under the R-symmetry, which nicely illustrates the power of this compactification inspired mnemonic.

Before closing, it may be worth noting that not *all* supersymmetric theories can be obtained from toroidal compactification of higher dimensional theories. The classic example of this is the Type IIB supergravity theory in ten dimensions. Starting from the (unique) $\mathcal{N} = 1$ theory in 11 dimensions with 32 supercharges arranged in a Majorana fermion, we can of course compactify on S_1 to obtain a ten dimensional theory. This theory has $\mathcal{N} = 2$ Majorana-Weyl generators of supersymmetry, each housing 16 supercharges. All the tricks we developed above can be used to work out the particle content, and the resulting theory (called Type IIA supergravity) is a perfectly legitimate supersymmetric theory. It is not the only theory of supergravity in ten dimensions, however. While the two Majorana-Weyl generators of Type IIA have opposite chirality, we can just as well construct a theory in which the two Majorana-Weyl generators have the same chirality. This theory earns the imaginative title Type IIB. Among the assortment of fields in the Type IIB supergravity multiplet, there exists a particularly interesting 4-form field, A_4. This field is interesting because its field strength is self-dual ($F_5 = dA_4 = \star F_5$), a constraint which is impossible to impose in any suitable Lagrangian formulation. The fact that there is no way to write down a Lorentz invariant action for Type IIB supergravity is suggestive of the futility of arriving at this theory via dimensional reduction. Given a valid 11 dimensional Lagrangian, what would we shoot for? Interestingly, both the IIA and IIB supergravity theories play a prominent role in string theory, where they arise as low energy effective field theories.

5. The Moral of the Story

In some important ways, we have exhausted extended supersymmetry. We know in detail how the supersymmetry algebra works, and how its representations are realized. Specifically, we saw that when $\mathcal{N} > 1$, central charges can appear, and their precise relation to the BPS bound governs the structure of the supermultiplets. Moreover, we carefully dissected the Dirac algebra and arrived at a comprehensive categorization of spinors in various dimensions. Importantly, we saw that in general the Dirac spinor transforms as a reducible representation of the Lorentz group, and can be further subdivided based on its chirality and reality properties. Better yet, by combining many of our earlier observations with the notion of the number of independent supercharges as a SUSY theory "invariant", we succeeded in decomposing the $\mathcal{N} = 1$ gauge multiplet in ten dimensions into the gauge multiplet of $\mathcal{N} = 4$ in four via toroidal compactification. Globally, we learned important lessons about Clifford algebras, and regained considerable familiarity with the "highest weights" method of multiplet construction.

In other important ways, there is still much to be done. Enumerating the particle content of generic extended SUSY supermultiplets is really only half the battle. Conspicuously absent from our discussion is an algorithm for writing SUSY invariant Lagrangians for these theories. Basically, we've got all the ingredients, but no recipe to tell us what to do with them. Of course this is no oversight. In the $\mathcal{N} = 1$ theory, we are spoiled by the availability of superfields which transform as irreducible representations of the SUSY algebra. These superfields permit an off-shell formulation of the theory, and allow one to create manifestly supersymmetric Lagrangians without much effort. Unfortunately, generalizing this program to arbitrary theories of extended supersymmetry can be hopelessly complex. Nonetheless, explicit component form Lagrangians for many of these theories do exist, and they are more than suitable for calculations. One way or the other, the formalities of extended supersymmetry are behind us, and all manner of interesting applications await.

References

[1] R. Hagg, J. Lopuszański, and M. Sohnius, *Nuclear Physics* B88 (1973)

[2] J. Maldecena, *The Large N Limit of Superconformal Field Theories and Supergravity*; arxiv:hep-th/9711200

[3] M. Srednicki, *Quantum Field Theory*; Cambridge (2007)

[4] A. Zee, *Quantum Field Theory in a Nutshell*; Princeton (2003)

[5] J. Lykken, *Introduction to Supersymmetry*; arxiv:hep-th/9612114

[6] E. D'Hoker and D. Freedman, *Supersymmetric Gauge Theories and the Ads/CFT Correspondence*; arxiv:hep-th/0201253

[7] D. Olive and P. West *Duality and Supersymmetric Theories*; Cambridge (1999)

[8] J. Polchinski *String Theory Vol. II* Cambridge (1998)

[9] E. Kiristas, *String Theory in a Nutshell*; Princeton (2007)

[10] M. Nakahara, *Geometry, Topology and Physics*; (2003)