A Supplement to *Strings*

Abstract

The following pages are intended to compliment my presentation *Strings from Scratch*. Although I have tried hard in the presentation to convey some important features of string theory in a way that is both accesible and (hopefully!) aesthetic, the presentation format makes it awkward to develop concepts with much rigor. Here I work through many of the important results outlined in the talk, and in doing so provide a more complete picture of the physics of strings.

String Actions

Of fundamental importance to succesfully calculating in string theory is a command of the string action. As an initial pass, lets write down the Polyakov action and obtain the equations of motion is describes:

$$S_p = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma\,\sqrt{-h}\,h^{\alpha\beta}\partial_\alpha X\cdot\partial_\beta X \tag{1}$$

here, $h^{\alpha\beta}$ is the world-sheet metric, while the string coordinates are understood to be contracted with the flat 2-D Minkowski metric. The world-sheet metric has the properties $h^{\alpha\beta} = (h^{-1})_{\alpha\beta}$ so $h^{\alpha\beta}h_{\alpha\beta} = 2$, and I have defined (as is customary) $h \equiv \det h_{\alpha\beta}$.

Varying the Polyakov action must be done for both the string coordinates and for the world-sheet metric. The former is the easier of the two, and serves as a nice warm-up. Since the action is clearly symmetric in the string coordinate X_μ, the variation of the action is twice the variation with respect to one of the coordinates in the scalar product $X\cdot X$,

$$\delta S = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma\,\sqrt{-h}\,h^{\alpha\beta}\partial_\alpha(\delta X^\mu)\partial_\beta X_\mu \tag{2}$$

The goal here is always the same—we want no derivatives acting on the variation, which can be accomplished by writing the derivative like

$$\partial_\alpha(\delta X^\mu)\partial_\beta X_\mu = \partial_\alpha(\delta X^\mu\partial_\beta X_\mu) - \delta X^\mu\partial_\alpha\partial_\beta X_\mu$$

so that the variation becomes

$$\delta S = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma\,\sqrt{-h}\,h^{\alpha\beta}\left[\partial_\alpha(\delta X^\mu\partial_\beta X_\mu) - \delta X^\mu\partial_\alpha\partial_\beta X_\mu\right] \tag{3}$$

In other words, I have simply integrated by parts. As always, this has led to a surface term, which is the first in the brackets above. Looking at this term alone, it is clear that it must vanish independently, so

$$\delta S_{boundary} = 0 = -\frac{1}{2\pi\alpha'}\int d\tau\left[\delta X^\mu\partial_\sigma X_\mu\right]\Big|_{\sigma_1}^{\sigma_2} - \frac{1}{2\pi\alpha'}\int d\sigma\left[\delta X^\mu\partial_\sigma X_\mu\right]\Big|_{\tau_1}^{\tau_2} \tag{4}$$

Because we require that the variation at the initial and final times be 0, the second term vanishes trvially. The first, however, gives us our boundary conditions for σ. Just as was illustrated on slide 13, this term can be made to vanish so long as one of the following conditions is met:

$$\begin{aligned} X_\mu(\sigma_1) &= X_\mu(\sigma_2) && \text{(Periodic)} \\ X'_\mu &= 0 && \text{(Neumann)} \\ X_\mu(\sigma = \sigma_1, \sigma_2) &= c && \text{(Dirichlet)} \end{aligned}$$

The remaining term in the variation of the action that must vanish is

$$\delta S_{EOM} = \frac{1}{2\pi\alpha'} \int d\tau\, d\sigma\, \delta X^\mu \partial_\alpha(\sqrt{-h}\, h^{\alpha\beta}\partial_\beta X_\mu) = 0$$

since this is assumed to be true for all variations δX^μ one can see that it must be that

$$\partial_\alpha\left(\sqrt{-h}\, h^{\alpha\beta}\partial_\beta X_\mu\right) = 0 \tag{5}$$

As it stands, it is hard to get much information out of equation (5). To put it in a more recognizable form, it will be necessary to pick a convenient gauge. As I suggest on slide 13, for an open string whose worldsheet is described by an infinite strip free of topological obstructions, one can use the conformal gauge. This gauge allows one to fix the world-sheet metric such that $h_{\alpha\beta} = \eta_{\alpha\beta}$, or

$$h_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

The determinant of this matrix is easy, $h = -1$, and we can now systematically write out the above equation of motion. Notice that all off diagonal terms in the metric vanish, so we get

$$\partial_\alpha\left(\sqrt{-h}\, h^{\alpha\beta}\partial_\beta X_\mu\right) = \eta^{\alpha\beta}\partial_\alpha\partial_\beta X^\mu = 0$$

or more familiarly,

$$\ddot{X}^\mu - X''^\mu = 0 \tag{6}$$

Which is of course just the wave equation! Next we must try a bit harder to obtain the equations of motion the correspond to the variation of the worldsheet metric. Solving Zwiebach's "Quick Calculation" 21.5 (trivial) one can show that given a 2×2 matrix A,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad \det A = a_{11}a_{22} - a_{12}a_{21}$$

and

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}, \qquad \delta A = \begin{pmatrix} \delta a_{11} & \delta a_{12} \\ \delta a_{21} & \delta a_{22} \end{pmatrix}$$

from which it is easy to see that

$$\delta \det A = a_{22}\delta a_{11} + a_{11}\delta a_{22} - a_{21}\delta a_{12} - a_{12}\delta a_{21} = \det A\, \mathrm{Tr}(A^{-1}\delta A)$$

not surprisingly, this generalizes to larger square matrices as well, but that doesn't matter here. In the present case, take

$$A = h_{\alpha\beta} \qquad \text{so} \qquad \det A = h\,, \quad A^{-1} = h^{\alpha\beta}$$

which, when inserted into the identity above, gives

$$\delta h = h h^{\alpha\beta} \delta h_{\alpha\beta}$$

The reason I am working through this is because the action has terms like $\sqrt{-h}h^{\alpha\beta}$, if you recall. To put this in a more useful form, remember that contracting $h_{\alpha\beta}$ with its inverse is equal to 2, so

$$\delta(h_{\alpha\beta}h^{\alpha\beta}) = 0 = \delta h_{\alpha\beta}h^{\alpha\beta} + h_{\alpha\beta}\delta h^{\alpha\beta}$$

Using the above relationship, and the fact that $h_{\alpha\beta}$ is symmetric, we are finally in a position to do some damage...

$$\begin{aligned}
\delta\sqrt{-h} &= -\frac{1}{2}\frac{\delta h}{\sqrt{-h}} \\
&= -\frac{1}{2}\sqrt{-h}\delta h^{\alpha\beta} h_{\alpha\beta}
\end{aligned}$$

returning to the Polyakov action action, note that

$$\begin{aligned}
\delta S &= -\frac{1}{4\pi\alpha'}\int d\tau\, d\sigma \left[(\delta\sqrt{-h})h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu + \sqrt{-h}\delta h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu\right] \\
&= -\frac{1}{4\pi\alpha'}\int d\tau d\sigma \sqrt{-h}\left[-\frac{1}{2}\delta h^{\alpha\beta}h_{\alpha\beta}h^{\gamma\delta}\partial_\gamma X^\mu \partial_\delta X_\mu + \delta h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu\right]
\end{aligned}$$

in case it is worth mentioning, the dummy indicies have been changed to avoid confusion—the $h^{\alpha\beta}$ only exists to contract the derivatives, so this is necessary. Again, the above expression must vanish for arbitrary variations $\delta h^{\alpha\beta}$. This makes it easy to see that

$$\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2}h_{\alpha\beta}h^{\gamma\delta}\partial_\gamma X^\mu \partial_\delta X_\mu = 0 \tag{7}$$

As per [1], the world-sheet energy momentum tensor is given by

$$T_{\alpha\beta} = -\frac{2}{t}\frac{1}{\sqrt{-h}}\frac{\delta S}{\delta h^{\alpha\beta}}$$

which, from (7) shows that for strings the tensor vanishes. As was done previously, this equation of motion is somewhat more interesting in the conformal gauge, where $h_{\alpha\beta} = \eta_{\alpha\beta}$ and

$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2}\eta_{\alpha\beta}(X'^\mu X'_\mu - \dot{X}^\mu \dot{X}_\mu)$$

The variation of the metric just gives the system of constraints

$$T_{\alpha\beta} = \begin{pmatrix} \dot{X}^2 + X'^2 & \dot{X}X' \\ \dot{X}X' & \dot{X}^2 + X'^2 \end{pmatrix} = 0 \tag{8}$$

One interesting thing to notice is that if one were to trace the energy-momentum tensor with the metric, he would find

$$\mathrm{Tr}\, T = \eta^{\alpha\beta}T_{\alpha\beta} = \dot{X}^2 + X'^2 - (\dot{X}^2 + X'^2) = 0$$

in this case, it is obvious that because we were able to choose the conformal gauge the trace vanished. The freedom to choose this gauge is a result of the reparametrization invariance of the world-sheet metric—the so-called Weyl invariance inherent in the string action.

Next I will see if I can show that "classically" the Polyakov action is identical to the Nambu-Goto action, given by

$$S_{NG} = -T \int d\sigma d\tau \sqrt{-\gamma} = -T \int d\sigma d\tau \sqrt{(\dot{X} \cdot X'^2)^2 - \dot{X}^2 X'^2} \tag{9}$$

where I have used the fact that the induced metric is defined by

$$\gamma_{\alpha\beta} \equiv \eta_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu \tag{10}$$

Now the vanishing of the energy-momentum tensor allows one to write

$$\partial_\alpha X^\mu \partial_\beta X_\mu = \frac{1}{2} h_{\alpha\beta} h^{\gamma\delta} \partial_\gamma X^\mu \partial_\delta X_\mu \tag{11}$$

this is encouraging, since the quantitiy on the LHS of (11) is simply the induced metric $\gamma_{\alpha\beta}$ as defined by (10). Interestingly enough, this expression shows that the induced metric is proportional to the world-sheet metric *at each point* on the world-sheet. We can now attempt to manipulate (11) until it looks like the Polyakov action. Start with

$$\gamma_{\alpha\beta} = \frac{1}{2} h_{\alpha\beta} h^{\gamma\delta} \gamma_{\gamma\delta}$$

and take the determinant of both sides, remembering that $h^{\gamma\delta} h_{\gamma\delta}$ is a scalar:

$$\begin{aligned} \det \gamma_{\alpha\beta} &= \frac{1}{4} \det h_{\alpha\beta} (h^{\gamma\delta} h_{\gamma\delta})^2 \\ \gamma &= \frac{h}{4} (h^{\gamma\delta} h_{\gamma\delta})^2 \end{aligned}$$

then multiply both sides by -1 and take the square root...

$$\sqrt{-\gamma} = \frac{1}{2} \sqrt{-h}\, h^{\gamma\delta} \partial_\gamma X^\mu \partial_\delta X_\mu$$

we may now relable the dummy indicies, multiply both sides by $-1/2\pi\alpha'$ and integrate over the world-sheet so

$$-\frac{1}{2\pi\alpha'} \int d\tau d\sigma \sqrt{-\gamma} = -\frac{1}{4\pi\alpha'} \int d\tau d\sigma\, \sqrt{-h}\, h^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta X_\mu \tag{12}$$

Equation (12) is one of my favorites. It shows that the Nambu-Goto string action and the Polyakov action are classically equivalent!

Before moving on and working out a few interesting results, I think it is worth writing a few comments about the symmetries of the actions. First I would like to explain what I meant in my slides when I said that the string actions are "manifestly reparametrization invariant". The idea will be to try to show that a term of the form

$$d\tau d\sigma \sqrt{-\gamma}$$

does not change if one were to use a new parametrization where $d\bar{\tau} \to d\bar{\tau}(\tau, \sigma)$ and $d\bar{\sigma} \to d\bar{\sigma}(\tau, \sigma)$. This is perhaps written in a rather confusing way, but I suspect what is meant will become clear in a second.

To do this right, remember that changing a measure always introduces a Jacobian, J such that

$$d\tau d\sigma = |J| d\bar{\tau} d\bar{\sigma} \qquad \text{where} \qquad J = \begin{pmatrix} \frac{\partial\tau}{\partial\bar{\tau}} & \frac{\partial\tau}{\partial\bar{\sigma}} \\ \\ \frac{\partial\sigma}{\partial\bar{\tau}} & \frac{\partial\sigma}{\partial\bar{\sigma}} \end{pmatrix}$$

Furthermore, if $|\bar{J}|$ describes the inverse transform, then it must be that $|J||\bar{J}| = 1$. Now note that, as defined on slide 8,

$$-ds^2 = \eta_{\mu\nu} = dX^\mu dX_\mu = \gamma_{ij} d\xi^i d\xi^j$$

this length certainly can not depend on the parametrization, so (with the help of the chain rule)

$$\gamma_{ij} d\xi^i d\xi^j = \bar{\gamma}_{mn} \frac{\partial \bar{\xi}^m}{\partial \xi^i} d\xi^i \frac{\partial \bar{\xi}^n}{\partial \xi^j} d\xi^j$$

From the above it is easy to see that

$$\gamma_{ij} = \bar{\gamma}_{mn} \frac{\partial \bar{\xi}^m}{\partial \xi^i} \frac{\partial \bar{\xi}^n}{\partial \xi^j} = \bar{\gamma}_{mn} \bar{J}_{mi} \bar{J}_{nj}$$

with an effort no greater than several lines of algebra (see for example [2], who uses a nice trick) it is straightforward to construct a relationship between $\gamma \equiv \det(\{\gamma_{ij}\})$, like

$$\gamma = \bar{\gamma}(\det \bar{J})^2$$

Now we are well poised to demonstrate what is meant by *reparametrization invariance*. Notice that the measure transforms like

$$d\sigma d\tau \sqrt{-\gamma} = d\bar{\sigma} d\bar{\tau}\, |J|\, \sqrt{-\bar{\gamma}}\, |\bar{J}| = d\bar{\sigma} d\bar{\tau} \sqrt{-\bar{\gamma}} \tag{13}$$

In other words, the measure does not change under a reparametrization of the world-sheet. This feature of string actions is often referred to as *diffeomorphism invariance* in the literature.

There is one more symmetry of the string action that deserves mention. This is the powerful *Weyl invariance* that allowed us to exploit merits of the conformal gauge and work with a traceless energy-momentum tensor.

Once you know what this symmetry looks like, it is very simple to show that it is respected in the Polyakov action. This is an invariance of the action under local rescaling of the world-sheet *metric*, ie.

$$h_{\alpha\beta} \to e^{\omega(\sigma,\tau)} h_{\alpha\beta}$$

to confirm that this is a good symmetry, it is clear that we must only check what happens to the term $\sqrt{-h}\, h^{\alpha\beta}$. Since $h^{-1}_{\alpha\beta} = h^{\alpha\beta}$, it follows that

$$h^{\alpha\beta} \to e^{-\omega(\sigma,\tau)} h^{\alpha\beta}$$

From this we obtain

$$\det h_{\alpha\beta} \to e^{2\omega(\sigma,\tau)} \det h_{\alpha\beta} \qquad \text{so} \qquad \sqrt{-h} \to e^{\omega(\sigma,\tau)} \sqrt{-h}$$

and thus

$$\sqrt{-h}\, h^{\alpha\beta} \to \sqrt{-h}\, e^{\omega(\sigma,\tau)} e^{-\omega(\sigma,\tau)}\, h^{\alpha\beta} = \sqrt{-h}\, h^{\alpha\beta} \tag{14}$$

At long last I think I have contented my self with the preliminary problems and I can try my hand at putting the above results to good use. In what follows, I will change pace a bit and work out some interesting results using the previously investigated features of the string action ...

String Stuffs

The following three worked examples offer a quick glance at a few fundamental results in string theory. I have tried to choose problems that address distinct and interesting aspects of the theory, from classical limits of relativistic actions to string interactions. All of three of these puzzles come from standard string theory texts, and are cited accordingly.

String Limits (Polchinski 1.1)

Here one must find the non-relativistic limits of the point-particle action and the Nambu-Goto string action. To make this problem a bit more straightforward, I have chosen to work in the static gauge where $X^0 = \tau$ in both cases:

For the relativistic point particle, we have

$$S = -m \int_\tau d\tau \sqrt{-\dot{X}^\mu \dot{X}_\mu} \qquad \text{(see slide \#5)}$$

in the static guage, $x^\mu = (\tau, x^1, x^2, x^3, \ldots, x^d)$ so

$$\begin{aligned} \dot{X}^\mu &= (1, v^1, v^2, \ldots, v^d) \\ \dot{X}^\mu \dot{X}_\mu &= -1 + v_1^2 + v_2^2 + \ldots + v_d^2 \\ &= -1 + v^2 \end{aligned}$$

in the non-relativistic limit $v \ll 1$, one expands $\sqrt{1-v^2} \approx 1 - \frac{1}{2}v^2$ which leaves

$$S \approx \int dt \, \{\frac{1}{2}mv^2 - m\} \tag{15}$$

Notice that the action contains contributions from the classical kinetic energy as well as the particle's the rest mass.

For the Nambu-Goto string action, we start from

$$S = -\frac{1}{2\pi\alpha'} \int d\tau d\sigma \, \sqrt{-\gamma}$$

and construct the matrix $\{\gamma_{\alpha\beta}\}$ using the static gauge condition $X^\mu = (\tau, \sigma, X^i)\ldots$

$$\{\gamma_{\alpha\beta}\} = \begin{pmatrix} -1 + \dot{X}^i \dot{X}_i & \dot{X}^i X_i' \\ X^{i'} \dot{X}_i & 1 + X^{i'} X_i' \end{pmatrix}$$

We need to take the determinant of this matrix, which is simplified somewhat by noting that the only terms that contribute will be those that are second order or less in the velocity fields. Accordingly, we keep

$$\gamma \approx -1 + \dot{X}^i \dot{X}_i - X^{i'} X_i'$$

This in turn means

$$\begin{aligned} \sqrt{-\gamma} &\approx \sqrt{1 - \dot{X}^i \dot{X}_i + X^{i'} X_i'} && (16) \\ &\approx 1 - \frac{1}{2}\dot{X}^i \dot{X}_i + \frac{1}{2}X^{i'} X_i' && (17) \end{aligned}$$

so

$$S \approx -\frac{1}{2\pi\alpha'} \int d\tau d\sigma \{1 - \frac{1}{2}\dot{X}^i \dot{X}_i + \frac{1}{2}X^{i'} X_i'\} \tag{18}$$

In analogy with the relativistic point particle, we see that the term

$$-\frac{1}{2\pi\alpha'} \int d\tau d\sigma = -m \int d\tau$$

so long as $m = lT$ where $l = \int d\sigma$ and T is the string tension. The second term represents the transverse kinetic energy, while the last is analogous to the potential energy in a non-relativistic string, ie. $U_{nr} = T/2(\partial y \partial x)^2$.

String Endpoints (Zwiebach 6.5)

Here is a somewhat surprising demonstration of the fact that string endpoints travel at the speed of light. I will show this in two different ways. . .

As you might recall, when I derived the string equations of motion in the conformal gauge, varying the metric led to the so-called *Viraroso constraints*,

$$T_{00} = T_{11} = \dot{X}^2 + X^{'2} = 0 \tag{19}$$

$$T_{01} = T_{10} = \dot{X} \cdot X' = 0 \tag{20}$$

now if the string endpoints are moving at all, it must be that the string is characterized by Neumann boundary conditons, and so at the ends $X' = 0$. We then have the relation $\dot{X}^2 = 0$. This equality is also found when working in the light cone gauge. As was shown previously, one can write

$$\dot{X}^2 = -1 + v^2 = 0$$

where $v = \partial_t X$. The solution is both wonderful and trivial, one finds $v = 1$. Reinserting appropriate dimensions,

$$v_{end} = c \tag{21}$$

The other way to see this is somewhat less direct, but as it is the way Zwiebach recommends in the problem, I will include it as well. Starting from the Lagrangian density for the Nambu-Goto string action,

$$\mathcal{L} = -T\sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2}$$

we can compute the momenta like

$$P^{\sigma}_{\mu} = \frac{\partial \mathcal{L}}{\partial X'} = -T \, \frac{(\dot{X} \cdot X')\dot{X}_\mu - \dot{X}^2 X'_\mu}{\sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2}}$$

so

$$P^{\sigma}_{\mu} P^{\mu\sigma} = T^2 \dot{X}^2 \, \frac{\dot{X}^2 X'^2 - (\dot{X} \cdot X')^2}{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2} \tag{22}$$

$$= -T^2 \dot{X}^2 \tag{23}$$

rewriting the boundary conditions in terms of these momenta recasts the Neumann boundary condition as $P^{\sigma}_{\mu} = 0$, so once more

$$\dot{X}^2 = 0 = -1 + v^2$$

or

$$v_{end} = c$$

which of course is exactly what was found above, see (21).

String Interactions (Zwiebach 22.7)

This is the problem I was most excited to solve from the start. Here we consider a four open string interaction in the special case where the incident strings have the same momenta—a sort of world-sheet center of mass frame. The picture here would look a bit like figure 1. Notice that

Figure 1: World-sheet diagram of a 4-string open tachyon interaction in which all incident and scattered strings carry the same momenta (ie. $p_1^+ = p_2^+ = p_3^+ = p_4^+ = 2\pi\alpha' p^+$)

unlike the general case where $p_1^+ \neq p_2^+ \neq p_3^+ \neq p_4^+$ in this interaction the parameter T can never be negative. This is intuitively clear by examining figure 1 . When $T_1 = T_2$, the 'slits' coincide and it no longer makes sense to measure the distance between them.

Here I will explicitly calculate the modulus, and show that in this (special) case, all the important features discussed on slide # 24 are present as they must be.

Following the flow chart on slide #21, it appears as though the first order of business will be to write the appropriate conformal map. Using the Schwarz-Christoffel map one finds

$$\begin{aligned} \theta_{P_1} = \theta_{P_2} &= \theta_{P_3} &= \theta_{P_4} = \pi \\ \theta_{Q_1} = &\quad \theta_{Q_2} &= -\pi \end{aligned}$$

which gives

$$\frac{dw}{dz} = \frac{A}{z}(z - x_1)\frac{1}{z-\lambda}\frac{1}{z-1}(z - x_2) \tag{24}$$

to make use of this, it will be useful to write this in a partial fractional decomposed form. This makes it easier to integrate the result, and provides a convenient form for comparing to the string 'Feynman rule'. This is done for a general four open string interaction in [2], and it won't contribute much to write the steps here. The result is

$$\frac{dw}{dz} = A\left(\frac{x_1x_2}{\lambda z} + \frac{(\lambda - x_1)(\lambda - x_2)}{\lambda(z-\lambda)(\lambda-1)} + \frac{(1-x_1)(1-x_2)}{(z-1)(1-\lambda)}\right) \tag{25}$$

To obtain an system of equations, we compare the above to the differential equation obtained from the string 'Feynman rule'. Starting from P_1 and going clockwise around the degenerate polygon, we increase σ twice, then decrease around P_3. The answer is

$$w(z) = -2\alpha' p^+ \ln z - 2\alpha' p^+ \ln(z - \lambda) + 2\alpha' p^+ \ln(z - 1)$$

Differentiating this expression and matching terms with (25) allows one to construct a system of equations. Again, I will spare you the algebra and write down the result. Taking ratios of the relationships eliminates the unknown constant A and simplifies the math. This gives

$$1 = \frac{(\lambda - x_1)(x_2 - \lambda)}{(1-\lambda)x_1x_2} \tag{26}$$

$$1 = \frac{(1 - x_1)(x_2 - 1)}{(1-\lambda)x_1x_2} \tag{27}$$

To solve this system it is obvious that we are short one equation. To remedy this, we use the constraint that the interaction time is given by $T = T_1 - T_2$. This, coupled with the results on slide #23 (specifically $T_1 = \Re w(x_1)$ and $T_2 = \Re w(x_2)$) plus a bit of tedious algebra, leaves

$$\frac{T}{2\alpha' p^+} = \ln \frac{x_2}{x_1} + \ln \frac{x_2 - \lambda}{\lambda - x_1} + \ln \frac{1 - x_1}{x_2 - 1} \tag{28}$$

Solving the system described by (26 –28) for the modulus, λ can be done many ways. After about a page of algebra, one finds

$$\lambda(T) = \frac{4e^{\frac{T}{4\alpha' p^+}}}{(1 + e^{\frac{T}{4\alpha' p^+}})^2} \tag{29}$$

I was very excited to obtain this result, because it has every property it should:

$$\begin{aligned} \lambda \to 0 \quad &\text{as} \quad T \to \infty \\ \lambda \to 1 \quad &\text{as} \quad T \to 0 \end{aligned}$$

what's (still) amazing about this result is that as T takes on every allowed value, λ ranges monotonically from 1 to 0. Just as before, since a Riemann surface with four ordered punctures *also* has an $\mathcal{N}_4$ moduli space, we can say that the string diagrams produce the moduli space of all such surfaces. The question also asks one to confirm that λ is a monotonic function of T. This should be obvious, since the only maxima or minima occur at the endpoints $T = 0, \infty$, but in case it needs more of an explanation I have included a plot in figure 2.

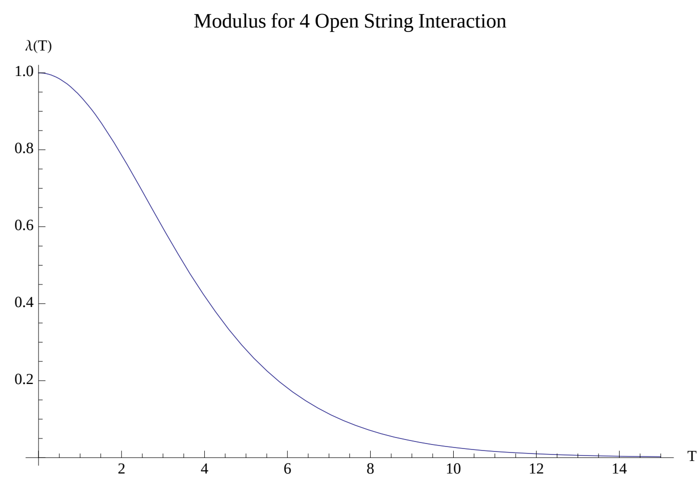

Figure 2: The modulus describing this 4-string open tachyon scattering as a function of time.

As a concluding remark, it is interesting to note that for four open tachyon scattering we describe the process with the Venziano amplitude. As described on slide #25, this amplitude is easily integrated and results in a mass spectrum identical to that expected for the strings:

$$M^2 = \frac{1}{\alpha'}(n - 1)$$

The following resources were helpful in compiling this supplement to my survey of string theory. They are a small subset of those cited on the slides...

References

[1] Becker, Becker, and Schwarz: *String Theory and M-theory*; Cambridge University Press (2007)

[2] B. Zwiebach: *A First Course in String Theory;* Cambridge University Press (2004)

[3] J. Polchinski: *String Theory*; Cambridge University Press (2001)