Preprint typeset in JHEP style - HYPER VERSION

Kinky Solutions to Quantum Field Theories: An Overview of Hedgehogs and Vortices

Christopher Rosen

University of Colorado, Boulder
E-mail: christopher.rosen@colorado.edu

Abstract: Although the perturbative approach to understanding the contents of a field theory can be remarkably successful, it is nonetheless inextricably bound to the theory's weakly coupled regime. Not surprisingly, field theories are capable of supporting a rich structure of non-perturbative phenomena, which no Feynman diagram can describe. Here we explore one breed of solution, known as the soliton, whose existence is independent of the value of the coupling constant. After realizing these modes in simple theories with broken symmetries, we will turn our attention to understanding the consequences of their existence.

Contents

1. The Stuck Mattress

As it turns out, field theories can be much more than a collection of sticks and loops. The weakly coupled, perturbative regime of a quantum field theory is well described by the Feynman calculus, but these powerful techniques are hopelessly inadequate for deciphering strongly interacting systems. Whats more, by probing only the weakly interacting field excitations, modes characterized by a non-analytic dependence on the coupling are utterly inaccessible. More precisely, consider a theory with coupling λ. If this theory contains a propagating mode with energy $E(\lambda) \propto 1/\lambda$, we will never find it in perturbation theory (since no Taylor series for $E(\lambda)$ exists about $\lambda = 0$).

In section 2, we will find modes with exactly this property in a simple field theory with spontaneously broken symmetry. These exotic solutions will be purely topological, dependent on non-trivial maps from the boundary of spacetime to the degenerate vacuua. Interestingly, these solitons will turn out to be non-dissipative. In the "spring mattress model" of quantum field theory, such modes are like a region of stuck springs that can propagate in unison, but never disperse.

Increasing the dimensionality of spacetime, and the number of fields that live in it, increases the possibility for interesting (non-trivial) maps between spatial infinity and the vacuum. These possibilities are enumerated by the so-called homotopy group, which we will briefly introduce. Among the solutions of interest, we will encounter the Nielsen-Olesen vortex, and the 't Hooft–Polyakov monopole.

The remainder of this review will be dedicated to deciphering the consequences of these strange solutions. This will allow us a quick glimpse into superconducters, charge quantization, and topology.

A Note on Conventions

This review assumes a "gravitational" metric with signature (−,+,+,+). When the stress tensor $T^{\mu\nu}$ appears, we always mean the "improved" stress tensor, which has the advantage of leaving many relevant symmetries explicit. For any action describing a matter field (i.e a non-gravitational action), S_m this tensor is defined by

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}} \tag{1.1}$$

where $g_{\mu\nu}$ is the metric of spacetime.

2. Simple Solitons in 1+1 Dimension

We begin our soliton search in the tried and true scalar theory described by the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{\lambda}{4}(\varphi^2 - v^2)^2 \tag{2.1}$$

As is well known, (2.1) has the amusing property of a degenerate ground state at $\varphi = \pm v$. Accordingly, any perturbative techniques we wish to apply to this theory should be formulated in terms of small fluctuations about either one of these two vacuua. Let's quickly review the consequences of such a perturbative analysis: one redefines the field, shifting by either vacuum expectation value (VEV), for example $\varphi(x) = \eta(x) + v$, so (2.1) becomes

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\eta\partial_\mu\eta - \lambda v^2\eta^2 - v\eta^3 - \frac{\lambda}{4}\eta^4 \tag{2.2}$$

Evidently, our theory of the scalar field ϕ contains in its spectrum a particle η with mass $m_\eta = v\sqrt{2\lambda}$.

For suitably small values of the coupling λ, the interactions of the η with itself are well described by applying the perturbation theory encoded in Feynman diagrams to (2.2). The η can propagate through space, and weakly interact with either two or three other η bosons. Is this the end of the story?

The answer, it happens, is a rather emphatic no. To defend this claim, we might search for field configurations that can not be classified as small oscillations about one vacuum or the other. These modes will be characterized by minimum energy configurations subject to some boundary conditions. Now the energy density in any scalar theory of the form $\mathcal{L} = -\frac{1}{2}(\partial_\mu\varphi)^2 - V(\phi)$ is governed by the stress tensor

$$T^{\mu\nu} = \partial^\mu\varphi\partial^\nu\varphi - g^{\mu\nu}\left[\frac{1}{2}\partial_\alpha\varphi\partial^\alpha\varphi + V(\varphi)\right] \tag{2.3}$$

as can easily be shown using (1.1).

The energy of some field configuration is just the integral of T^{00} over the volume of space. It will turn out to be interesting if we agree to work in one space and one time dimension, where this energy is given by

$$E = \int_{-\infty}^{\infty} dx\left[\frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}(\nabla\varphi)^2 + V(\varphi)\right] \tag{2.4}$$

First consider (2.4) in the context of the theory described by (2.1). To make life easier, we can choose to look for solutions that are time independent ($\dot{\varphi} = 0$). Because these solutions will be at rest, we relabel $E \to M$ where M is now the rest mass of the excitation. Since this theory is manifestly Lorentz invariant, if such modes exists we can always set them in motion via an appropriate Lorentz boost. Explicitly, we have

$$M = \int_{-\infty}^{\infty} dx \left[\frac{1}{2}(\nabla\varphi)^2 + \frac{\lambda}{4}(\varphi^2 - v^2)^2 \right] \tag{2.5}$$

In our hunt for germane solutions, we should certainly exclude the uninteresting configurations that are characterized by a divergent rest mass. Looking closely at (2.5), we note that both terms are positive, and thus *each* should give a finite contribution to the mass. Accordingly, to avoid a linear divergence in the second term, we must impose the boundary condition $\lim_{x\to\pm\infty} \varphi(x) = \pm v$.

There are thus two possibilities: option A, where both points of the spatial boundary are mapped to the same vacuum state ($\varphi(\pm\infty) = v$ *or* $\varphi(\pm\infty) = -v$), or option B, where each point of the spatial boundary is mapped to a different vacuum state (say $\varphi(\infty) = v$ *and* $\varphi(-\infty) = -v$). It is easy to see that there are no non-trivial field configurations that satisfy the criteria of option A. To this end, note that we can rewrite (2.5) like

$$\begin{aligned} M &= \int_{-\infty}^{\infty} dx \left[\frac{1}{2}\left(\nabla\varphi - \sqrt{2V(\varphi)}\right)^2 + \nabla\varphi\sqrt{2V(\varphi)} \right] \\ &= \int_{-\infty}^{\infty} dx \frac{1}{2}\left(\nabla\varphi - \sqrt{2V(\varphi)}\right)^2 + \int_{\varphi(-\infty)}^{\varphi(\infty)} d\varphi \sqrt{2V(\varphi)} \end{aligned} \tag{2.6}$$

Consider option A, with $\varphi(\pm\infty) = v$. From (2.6), it is obvious that the second term vanishes. This leaves us with the task of attempting to minimize the mass of the configuration while satisfying the boundary conditions. Because the remaining (first) term in positive, the smallest it could hope to be is zero. Apparently, the field configuration that will make it zero must solve the differential equation $\nabla\varphi = \sqrt{2V(\varphi)}$ such that $\varphi(\pm\infty) = v$. Of course this solution is easy to find—it's just the trivial vacuum configuration $\varphi(x) = v$!

This solution suffers three strikes: we already knew it existed, it's boring, and it's clearly not what we are after. Luckily there is option B, where the spatial boundary is mapped to unique vacuua. Repeating the analysis of the proceeding paragraph, we find that in this case the second term does not vanish, but instead contributes $M_s = \frac{2}{3}v^3\sqrt{2\lambda} = \frac{1}{3}(m_\eta^3/\lambda)$ to the rest mass of the excitation. If we can again find a way to satisfy the boundary conditions while making the first term disappear, M_s will then be the mass of our new, non-perturbative field configuration.

Again, this is not such a hard task. Now we solve $\nabla\varphi = \sqrt{2V(\varphi)}$, requiring $\varphi(\infty) = v$ and $\varphi(-\infty) = -v$. Integrating, one finds

$$\varphi(x) = v \tanh\left(\frac{m_\eta}{2}x\right) \tag{2.7}$$

where I have arbitrarily set the integration constant to 0, centering this solution on the origin. Equation (2.7) is to be our first example of a soliton, and as such it will pay to

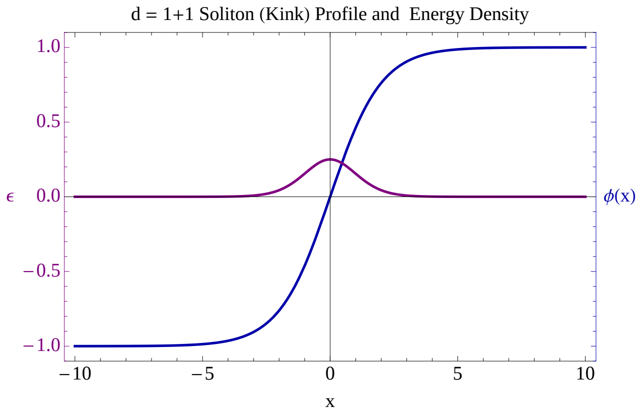

Figure 1: Kink profile and energy density plotted in units where $v = m_\eta = 1$.

spend a bit of time understanding it in some detail. The profile of this solution, and its energy density, are shown in figure 1. For obvious reasons, this configuration is known as a "kink". Evidently, the energy density is localized in a region $l_s \sim 1/m_\eta$. Interestingly, both the size of this soliton (l_k) and its mass (M_s) are controlled by the mass and quartic coupling in the theory of interacting η bosons. The fact that the mass of the kink is inversely proportional to the quartic coupling, λ, means we should not be surprised that these objects do not appear as dynamical modes in perturbation theory. Imagine studying the Lagrangian (2.2) in its weakly coupled regime. Here the kink is extremely heavy, and the corresponding theory is one in which these modes are effectively "frozen out". Thus these degrees of freedom will never appear in any Feynman diagram.

Beyond the relativistic field theory studied here, it is interesting to consider how a kink might arise in condensed matter systems as well. Consider the canonical example of an infinite "chain" of spins in the presence of an external magnetic field. At zero temperature, the spins align with the applied field. At sufficiently high temperature, the spins are disordered. We can quantify this behavior by studying the order parameter, $m(x)$, which in this case tells us about the variation of the magnetization across space. In the context of our field theory example above, we recognize $m(x) \to \langle\varphi(x)\rangle$. This identification immediately allows us to describe a kink in the spin chain: it would look like $m(x) = M$ for $-\infty < x < 0$ and $m(x) = -M$ for $0 < x < \infty$, where M is a constant. Such defects do indeed appear in condensed matter, often as "domain walls" in higher dimensional systems.

Before considering some more technical aspects of these soliton solutions, lets introduce one more. For this we will need the Sine-Gordon Lagrangian, which has the form

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{\alpha}{\beta^2}\left(1 - \cos\beta\varphi\right) \tag{2.8}$$

where α, β are real constants. The generic analysis performed above certainly applies to this scalar field theory as well. Turning the crank, we solve

$$\nabla\varphi = \sqrt{2\frac{\alpha}{\beta^2}\left(1 - \cos\beta\varphi\right)} \tag{2.9}$$

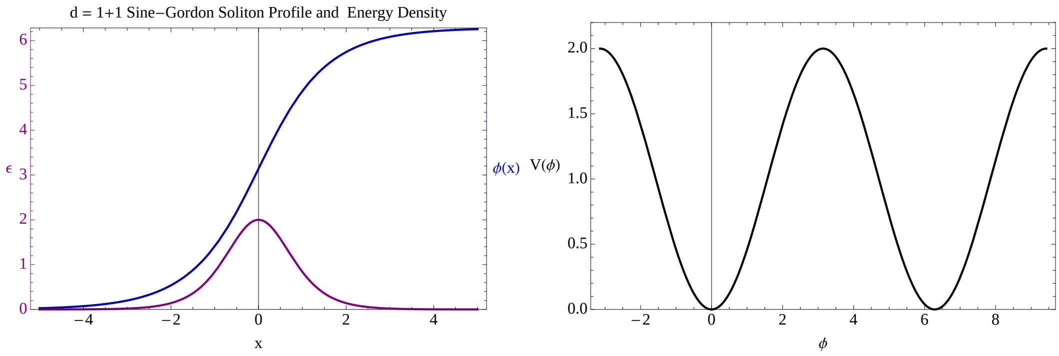

Figure 2: [Left] Sine-Gordon soliton profile and energy density plotted in units where $\alpha = \beta = 1$. [Right] The relevant part of the Sine-Gordon potential.

which is easily accomplished by integration. The result is a soliton of the form

$$\varphi(x) = \frac{4}{\beta} \arctan\left[\exp(x\sqrt{\alpha})\right] \tag{2.10}$$

which is shown in figure 2 superimposed atop its energy density. Also shown in figure 2 is the potential experienced by the scalar field. Note that just as for the kink, this mode interpolates between two distinct vacuua at the spatial boundary. Specifically, $\varphi(\infty) \to 2\pi$ and $\varphi(-\infty) \to 0$. We can of course choose *any* two vacuua to map the endpoints of space into—this corresponds to the ability to select different branches of the arctangent in (2.10).

Until now, we have ignored one fantastically important property that these solutions must have. In order to be interesting solutions in a dynamic theory, these defects should be robust against small perturbations in the field. The question of the stability of the soliton can be answered on many levels. Presently, we will understand this stability from the conservation of a topological current, as suggested by [2]. In section 3.1, we will revisit this problem in the context of homotopy groups. For an alternative explanation, the reader is referred to [3] where the question is rephrased in the language of a simple problem in quantum mechanics.

Note that commuting partial derivatives ensure the conservation of the current

$$J = n(\star \mathrm{d}\varphi) \to J^\mu = n\epsilon^{\mu\nu}\partial_\nu\varphi \tag{2.11}$$

in any flat space d= 1+1 scalar field theory. Here, n is an arbitrary constant we can use to normalize the charge as convenient. Conservation laws like (2.11) do not require a continuous symmetry of the Lagrangian like those of Noether. Instead, they are consequences of the topology of the manifold on which the fields live. In the present case, it is the existence of a map from the one-form gradient of the scalar field to a one-form current in two dimensions (the Hodge $\star$) that allows this current. It is easy to see that there exists analogues of (2.11) in d dimensions for any $(d-2)$-form fields. In any case, the conserved charge resulting from (2.11) will be given by

$$Q = n\int_{-\infty}^{\infty} \epsilon^{01}\partial_1\varphi = n\left[\varphi(\infty) - \varphi(-\infty)\right] \tag{2.12}$$

as per the fundamental theorem of calculus. For the φ^4 theory in its broken phase, choose $n = (1/2v)$ and for the Sine-Gordon theory $n = (1/2\pi)$. Then both of these configurations have $Q = 1$. Now consider the interaction of these soliton modes with a small fluctuation in the field (for example, the η boson in (2.2)). By definition, a small fluctuation is localized, vanishing at the boundary of space. From (2.12), we see that these perturbations must have charge $Q = 0$. Accordingly, there exists no way for a small fluctuation to "straighten" these one-dimensional defects (1+0 =1). As promised in the introduction, this solution is non-dissipative—once a kink, always a kink.

2.1 Bogomol'nyi and Derrick

There are a few more points of note worth discussing before we hunt for solitons in higher dimensional theories. First, observe that the two terms in (2.6) are manifestly positive. In the φ^4 theory, the second term in this expression integrates to $\frac{1}{3}(m_\eta^3/\lambda)$, and we find that the mass is bounded from below:

$$M_s \geq \frac{1}{3}(m_\eta^3/\lambda) \tag{2.13}$$

If we rescale the conserved current like $n \to n' = m_\eta(v/3)$ we can rewrite this inequality as

$$M_s \geq |Q| \tag{2.14}$$

A constraint on the mass of this sort is generically called a *Bogomol'nyi bound*, and is a feature common to descriptions of solitons we will encounter in this review. The Bogomol'nyi bound is a nontrivial relationship between a conserved charge and the mass of the excitation. Clearly, both of our 1+1 dimensional soliton solutions (2.7) and (2.10) saturate this bound. Indeed, we indirectly used this fact to help identify them. Such solutions are called *Bogomol'nyi-Prasad-Sommerfeld* (BPS) solitons. In theories with extended supersymmetry ($\mathcal{N} > 1$), BPS states are particularly interesting because they lead to shortened multiplets (see e.g. [4]).

The second point will serve as a warmup for the sections to come. Imagine we extend our soliton search to 2+1 dimensions. Our experience in section 2 tells us that in order to have non-trivial solutions, we need a non-trivial vacuum. In two dimensions, spatial infinity is topologically equivalent to a circle, S^1. Is there some field theory whose vacuua also have this topology? Of course the answer is yes—one easy example is the complex scalar field in its broken phase,

$$\mathcal{L} = -\partial_\mu\varphi^\dagger\partial^\mu\varphi - \frac{\lambda}{4}\left(\varphi^\dagger\varphi - v^2\right)^2 \tag{2.15}$$

clearly the VEV of this field is given by $\langle\varphi^\dagger\varphi\rangle = v^2$, the vacuum lies on a circle in field space with radius v.

Just as before, we can look for time independent minimum energy field configurations in

$$M = \int_{-\infty}^{\infty} d^2x \left[|\vec{\nabla}\varphi|^2 + \frac{\lambda}{4}(\varphi^\dagger\varphi - v^2)^2\right] \tag{2.16}$$

This time, we avoid a quadratic divergence by enforcing $\varphi(r=\infty,\theta)=ve^{i\theta}$. That takes care of the second term in (2.16), but watch what happens to the first. In polar coordinates, $d^2x \to rdrd\theta$ and $\nabla\varphi = \hat{r}\partial_r\varphi + \hat{\theta}(1/r)\partial_\theta\varphi$. At infinity, the first term is $|\nabla\varphi|^2 \to v^2/r^2$, and (2.16) diverges logarithmically. This is to be expected, as a consequence of the famed "Derricks's Theorem" which asserts that there are no finite-energy, time-independent solitons localized in $d>1$ in a theory built only from scalar fields. To understand why this is the case[1], consider a theory of N scalar fields $\varphi_{i=1,2...,N}$ in $d=D+1$ dimensions. The appropriate Lagrangian is simply

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\varphi_i\partial^\mu\varphi_i - V(\varphi_i) \tag{2.17}$$

where we take $V(\varphi_i)\geq 0$. If $\varphi_i(x)$ is a soliton solution, then its energy will be given by $E=T+U$ where

$$T = \int d^Dx\,|\nabla\varphi_i|^2 \tag{2.18}$$

$$U = \int d^Dx\,V(\varphi_i) \tag{2.19}$$

and φ_i minimizes E. Under a scaling, $x\to x'=x/\alpha$, the measure transforms as $d^Dx \to \alpha^D d^Dx'$ and the gradient scales like $\nabla\to\alpha^{-2}\nabla'$. The energy is now given by

$$E(\alpha) = \alpha^{D-2}\,T + \alpha^D\,U \tag{2.20}$$

By definition, $\varphi_i(x')|_{\alpha=1}$ is the field configuration that minimizes the energy. Accordingly, it must be that

$$\partial_\alpha E(\alpha)\Big|_{\alpha=1} = 0 = (D-2)T + DU \tag{2.21}$$

For an arbitrary (non-trivial) potential that satisfies $V(\varphi_i)\geq 0$, we find that (2.21) can not be satisfied for *any* collection of scalar fields for $D\geq 2$ (since both T and U are manifestly positive). Apparently, there can exist no finite energy field configuration constructed from scalar fields, localized in more than one spatial dimension.

3. Further Down the Spiral: The Vortex in d = 2+1

At long last we are prepared to leave our linear world and upgrade to more spatial dimensions. Our first stop will be physics on a plane. Inspired by Derrick (section 2.1), we know that there will be no topologically interesting solutions in any theory we build exclusively from scalar fields. Loosely, the problem was that the kinetic term for our soliton configuration diverged, rendering the solution useless. A reasonable course of action might be to try adding some field that transforms non-trivially under the Lorentz group, and use this field to help nullify some of the infinities we encountered.

[1] The proof offered here follows problem 92.1 of [1], which is problem 1.1 in [5].

The simplest such choice is the vector, A_μ, which we already know can be used to gauge the global $U(1)$ symmetry present in (2.15). The corresponding theory, written in terms of the covariant derivative $D_\mu\varphi = (\partial_\mu - ieA_\mu)\varphi$, is

$$\mathcal{L} = -(D^\mu\varphi)^\dagger D_\mu\varphi - \frac{1}{4}\lambda\left(\varphi^\dagger\varphi - v^2\right)^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{3.1}$$

where $F_{\mu\nu}$ is the usual two-form field strength. Under an appropriate gauge transformation, we can always force $A_0 = 0$. In this case, time independent scalar field configurations have a kinetic energy given by

$$|D_\mu\varphi|^2 \to |\vec{D}\varphi|^2 = |\vec{\nabla}\varphi - ie\vec{A}\varphi|^2 \tag{3.2}$$

which suggests that a clever choice of $\vec{A}$ might be made to cancel the divergences we found below (2.16). At spatial infinity, $\varphi \to ve^{i\theta}$ so $\vec{\nabla}\varphi \to v(i/r)e^{i\theta}$, and the requirement $\vec{\nabla}\varphi \sim ie\vec{A}\varphi$ forces $\vec{A} \to (1/er)\hat{\theta}$ at infinity. Because the kinetic term for this gauge field goes like $|F|^2 \sim (1/r^4)$ at large r, this choice clearly renders the soliton energy finite. As pointed out in [1], these asymptotic field configurations correspond to local $U(1)$ transformations ($U = e^{i\theta}$) of the vacuum, $\langle\varphi\rangle = v$ and $\langle A_\mu\rangle = 0$. With this in mind, we take our soliton solutions to be of the general form

$$\varphi(r,\theta) = vf(r)e^{i\theta} \tag{3.3}$$

$$\vec{A}(r,\theta) = \frac{1}{er}g(r)\hat{\theta} \tag{3.4}$$

where $f(r)$ and $g(r)$ are undetermined functions fixed by minimizing the energy of the field configuration. These functions are subject to the boundary conditions $f(0) = g(0) = 0$ to keep the fields and their gradients well defined at the origin, and $f(\infty) = g(\infty) = 1$ to keep the energy of the soliton finite.

Even without explicitly solving for $f(r)$ and $g(r)$, we can learn quite a bit about the soliton configuration described by (3.3). As a consequence of this theory's $U(1)$ gauge symmetry, the action (3.1) is obviously invariant under the *global* $U(1)$ transformation $\varphi \to U\varphi = e^{i\eta}\varphi$ for constant η. Accordingly, there is a conserved Noether current readily computed in the standard way, which looks like

$$J^\mu = i\left[(D^\mu\varphi)^\dagger\varphi - \varphi^\dagger D^\mu\varphi\right] \tag{3.5}$$

and sources the electromagnetic sector of the theory, $\partial_\mu F^{\mu\nu} = J^\nu$. By construction, this current vanishes far from the origin, but the fields do not. We can easily compute the flux of the field strength by integrating the form F over S^2:

$$\Phi = \int_{S^2} F = \int_{S^2} dA = \int_{S^1} A_\mu dx^\mu \tag{3.6}$$

where we have used Stoke's Theorem to rewrite the flux as an integral of the one-form potential A over a circle at infinity. Using (3.4) with the appropriate boundary condition $g(\infty) = 1$, the integral is easy—we find $\Phi = (2\pi/e)$. Evidently, a weakly charged scalar field gives rise to solitons with large magnetic flux.

Interestingly, there is an ambiguity in the way we have constructed our fields (3.3) and (3.4). In order to keep the energy finite, we required $\varphi(r = \infty, \theta) = ve^{i\theta}$. Furthermore, we chose to study this theory for the important reason that there exists a map between spatial infinity, S^1_∞ and the space of vacuum configurations, S^1_v. Take the coordinate θ to designate points on S^1_∞ and use the coordinate α on S^1_v. Our map $m : S^1_\infty \mapsto S^1_v$ is a relationship that takes values of θ into values of α. Generically, our field at infinity takes the form $\varphi(r, \theta) \sim e^{i\alpha(\theta)}$. To keep the field regular ($\varphi(\theta) = \varphi(\theta+2\pi)$), we find $\alpha(\theta+2\pi) = \alpha(\theta)+2\pi n$, for $n \in \mathbb{Z}$. The idea is that we can loop around S^1_v n times as we circumnavigate S^1_∞ once—for $n > 1$ this map is onto, but not one-to-one. A simple map that preserves this property is $\alpha : \theta \mapsto n\theta$. With this in mind, we can write $\varphi(r = \infty, \theta) = ve^{in\theta}$ which reduces to our original ansatz for the special case $n = 1$.

Retracing our steps, we see that we can modify our earlier results to reflect this map, like

$$\varphi(r,\theta) = vf(r)e^{in\theta} \tag{3.7}$$

$$\vec{A}(r,\theta) = \frac{n}{er}g(r)\hat{\theta} \tag{3.8}$$

$$\Phi = \frac{2\pi n}{e} \equiv n\Phi_0 \tag{3.9}$$

Evidently, if we employ such a map from S^1_∞ to S^1_v our solitons will *necessarily* carry magnetic flux quantized in units of $\Phi_0 = (2\pi/e)$. This intriguing result was already anticipated by Nielsen and Olesen when they found these solutions in 1973 [6]. In fact, these authors were motivated by applications of Landau-Ginzburg theory (qualitatively (3.1) for time independent field configurations with $A_0 = 0$) to the theory of type II superconductors. To capture their line of reasoning, it is useful to consider our 2+1 dimensional theory embedded in one higher spatial dimension. In other words, our fields are located at points (t, r, θ, z), and we will assume that the physics is independent of t and z. It doesn't take much imagination to decide that our flux-*holes* have now grown into flux-*tubes* that connect z^+ and z^-, the spatial boundaries in the z direction.

Now it is a well known result from the theory of superconductors that interesting things happen when they are immersed in a magnetic field. Generically, superconducting materials (in their superconducting phase) expel magnetic fields from their interior, as described by the Meissner effect. When the applied field is weak, this occurs over a characteristic length called the London penetration depth. As the field increases, the superconductor will do one of two things: a *type I* superconductor will cease superconducting for some critical field strength. Above this value of the field, there is a region in which a *type II* superconductor will form flux tubes that carry the magnetic field through the material. In this state, a type II super conductor remains superconducting. As it turns out, these flux tubes transport quantized magnetic flux, and the current in the material circulates around them like a tornado. For this reason, these flux tubes are dubbed vortices. Shortly, we will see that our solutions also have this circulation, and we have already decided that they carry quantized magnetic flux—accordingly we will call our d = 2+1 solitons vortices as well.

Originally, Nielsen and Olesen were inspired by the flux tubes of type II superconductivity because they reminded them of the Veneziano model of hadron interactions. This

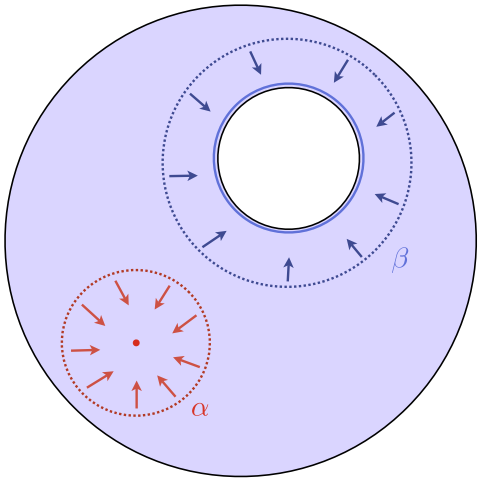

Figure 3: Illustration of the essential idea of a homotopy. Here we look at the image of the maps $\alpha, \beta : S^1 \mapsto N$ in N, the annulus (lavender). Although α and β are both loops in N, they belong to different equivalence classes (i.e. they are not homotopic) because α can be shrunk to a point, whereas β cannot. The set of equivalence classes forms the fundamental group, $\pi_1(N)$.

model, in its original form (and its current incarnation, string theory!) is a theory in which the fundamental degrees of freedom are "strings" as opposed to point particles. The Veneziano model had noteworthy experimental success at describing hadron phenomenology [7], before Quantum Chromodynamics emerged as its replacement. Nielsen and Olesen were trying to find a relativistic field theory that gave rise to these Veneziano strings, and trying to show that the strings/flux tubes obeyed a Nambu action (which is proportional to the area of the string world sheet). The idea of Nielsen-Olesen vortices as strings is anything but forgotten, appearing in various contexts even to this day [8].

3.1 Mathematical Interlude

At first glance, our insistence upon the existence of a map $\alpha(\theta)$ from spatial infinity to the vacuua of our theory may seem superficial. We found that such a map quantizes the magnetic flux spouting from our vortex, which is interesting, but so far unjustified. Indeed, our original assertion (that $\varphi^\dagger\varphi \to v^2$ as $r \to \infty$) only fixes the magnitude of $\varphi(r = \infty, \theta)$, and says nothing about what we should choose for our phase. Actually, there is an excellent reason for us to enforce the map between S^1_∞ and S^1_v—it will guarantee the stability of our solution! This surprising fact is a beautiful consequence of the homotopy group $\pi_1(S^1)$ which we will discuss now at no great length.

Briefly, a homotopy is an equivalence relation that identifies maps between two topological spaces. More specifically, given two spaces M and N, and two maps $m_{i=1,2} : M \mapsto N$, we say that $m_1 \sim m_2$ if the image of m_1 can be continuously deformed into the image of m_2 in N. This whole business is easily explained with a cartoon. Take $M = S^1$ a circle, and

N the annulus shown in figure 3. Two maps from M to N can be α and β, whose images in N are just loops. From the figure, it is obvious that there is something fundamentally different between the two maps. Under continuous deformation (i.e. without cutting or gluing) it is easy to see that α can be shrunk to a point in N. If α and β are to be in the same equivalence class, then we should be able to do this to β as well. Obviously this is impossible. As we contract β, we eventually run into the hole in the middle of the annulus, and can go no further. It is easy to see that this example supports many equivalence classes, each characterized by how many times the loop winds around the hole, and in what direction (clockwise or counterclockwise). It turns out that there is a well defined product between loops, and thus between equivalence classes. One can show [5] that these equivalence classes support a group structure, which is called the fundamental group, denoted $\pi_1(N)$. In so many words or less, the fundamental group catalogues all the topologically distinct maps from the circle to the space N.

The example in the preceding paragraph was chosen to evoke a connection to our Nielsen-Olesen vortex solutions. There too we had maps distinguished by their integer winding number, n. In the vortex case, we should talk about the fundamental group of S^1_v, the topological space equivalent to the scalar vacuum. Earlier, we wrote our map explicitly as $\alpha(\theta) = n\theta$. For $n = 0$, $\alpha = 0$ for all θ. This is the trivial map, where the entirety of S^1_∞ is mapped to a point in S^1_v. For $n = \pm 1$, we have $\alpha = \pm\theta$. This is the identity map, where each point in S^1_∞ is mapped to the "same" point in S^1_v, in either a clockwise or counterclockwise manner. From this, it is easy to deduce the fundamental group of S^1_v. Since loops in S^1_v look like $e^{in\theta}$, the product of two loops is a third: $e^{in\theta} \cdot e^{im\theta} = e^{i(n+m)\theta}$. Thus, we surmise that the fundamental group is the additive group of integers, $\pi_1(S^1_v) = \mathbb{Z}$.

Notice that the trivial map $n = 0$ corresponds to the identity element in $\mathbb{Z}$, 0. Since $\mathbb{Z}$ is a discrete group, arbitrary group elements are not continuously connected to the identity. In terms of our maps, there is no way to turn, say, the identity map into the trivial map. Intuitively, you can't take a rubber band off of a garden hose without scissors. This fundamental result, given to us by the fundamental group, guarantees the topological stability of our vortex solution. In other words, a stable vortex requires the existence of a map from spatial infinity to the space of vacuum configurations, and this map in turn forces the vortices to carry quantized magnetic flux.

3.2 A Vortex By Any Other Name

With these technicalities behind us, we can return to our vortex for a few final considerations. Just as we have done for the kink, it is straight forward to compute the energy of our vortex configuration by integrating T^{00} over space. Using (1.1), it is easy to compute the energy tensor

$$T^{\mu\nu} = (D^\mu\varphi)^\dagger D^\nu\varphi - g^{\mu\nu}\left[\frac{1}{2}(D^\mu\varphi)^\dagger D_\mu\varphi + V(\varphi^\dagger\varphi)\right] - \frac{1}{4}\left[g^{\mu\nu}F^2 - 4F^{\mu\alpha}F^\nu{}_\alpha\right] \tag{3.10}$$

which, using the fact $B^i = \frac{1}{2}\epsilon^{ijk}F_{jk}$, implies that for time independent solutions in temporal gauge ($A_0 = 0$),

$$E = \int d^2x \, T^{00} = \int d^2x \left[|\vec{D}\varphi|^2 + V(\varphi^\dagger \varphi) + \frac{1}{2} B^2 \right] \tag{3.11}$$

To find the minimal energy configuration, we require $\delta E = 0$, and insert the field ansatz (3.7) and (3.8) into the resulting equations of motion. The consequence is a rather unspectacular system of coupled, second order differential equations for the functions $f(r)$ and $g(r)$. In general, these equations can only be solved everywhere numerically for arbitrary values of the winding number, n, and the couplings λ, e, v. Fortunately, for our present purposes it will suffice to examine the solutions near the vortex core, where $\rho \equiv evr << 1$. In this limit, the equations of motion decouple, and reduce to

$$f'' + \frac{f'}{\rho} - \frac{n^2 f}{\rho^2} = 0 \tag{3.12}$$

$$g'' - \frac{g'}{\rho} = 0 \tag{3.13}$$

The goal is to solve these equations, subject to the boundary condition $f(0) = g(0) = 0$. This is no great task. The second of these, (3.13) is easily integrated to give $g(r) = m_\gamma^2 r^2$, where $m_\gamma = ev$ is the mass of the "photon". A computer has no trouble solving the first (3.12), the solution is $f(r) \sim m_\gamma^n \, r^n$.

Consider these near-core solutions for the special (but arbitrary) case $n = 1$. We can gain some valuable insight into our vortices by examining the behaviour of the conserved current (3.5) near $r = 0$. In Cartesian coordinates, we have $J^i = \frac{\partial x^i}{\partial r} J^r + \frac{\partial x^i}{\partial \theta} J^\theta$, or

$$J^x = -2y\sqrt{x^2+y^2}\left(x^2+y^2-1\right) + 2x(x^2+y^2) \tag{3.14}$$

$$J^y = 2x\sqrt{x^2+y^2}\left(x^2+y^2-1\right) + 2y(x^2+y^2) \tag{3.15}$$

where $x = x^1$, $y = x^2$ and all constants have been set to one. This current is shown in figure 4. Lo and behold, the charge is circulating around the soliton core! This somewhat satisfying observation, coupled with the fact that these solutions carry quantized magnetic flux (3.9), fully justifies our insistence on calling them vortices.

4. Monopoles!

Finally, we are prepared to examine the most famous topological defect of all. Fortunately for us, all the formal machinery has been developed in sections 2 and 3, so we are more or less on autopilot until it comes time to interpret our results...

We now turn to the construction of soliton solutions in d = 3+1. Specifically, our goal is to find a stable defect localized in three spatial dimensions. Our homotopic instincts tell us that the (stable) foundation on which we should build our solution is a nontrivial homotopy group, so what options exist? This group should characterize maps from spatial infinity to the space of vacuua. In d = 3+1, spatial infinity is topologically equivalent to the sphere, which we can denote S^2_∞. The problem, then, is to identify an interesting

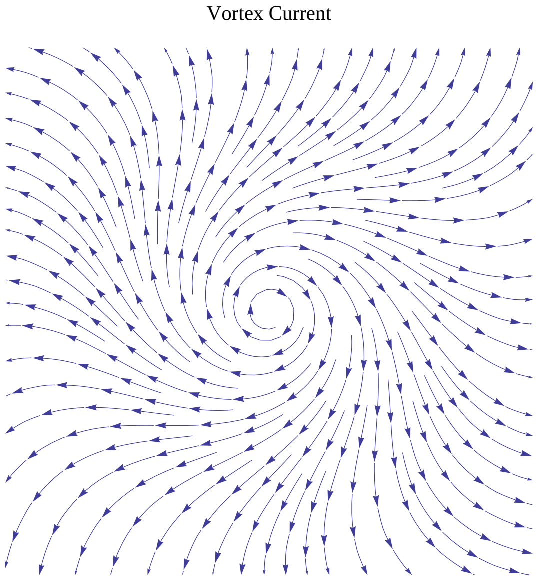

Figure 4: Behaviour of the conserved current, J^μ, near the vortex core. This circulating charge is a vortex hallmark.

topological space N such that $\pi_2(N) \neq 0$, and devise a theory whose vacuua share this topology.

It is easy to see that there are no non-trivial maps $m : S^2_\infty \mapsto S^1_v$, but there *are* interesting maps from S^2 to S^2. As an example, given two beach balls, we could map all the points on the surface of one into the north pole on the surface of the other (trivial), but we could also identify each point on one with the "same" point on the other (north pole $\mapsto$ north pole, etc.) which is the identity map. Indeed, it is not hard to demonstrate the well known math fact $\pi_2(S^2) = \mathbb{Z}$. As before, our defects will thus be characterized by an integer winding number, n. That these solutions will be stable is easily understood by the well known life fact that its impossible to remove the yolk from an egg without cracking the shell.

All this strongly suggests we look to theories with vacuua equivalent to S^2. It is easy to guess one such theory, that of three real scalar fields with the potential

$$V(\varphi) = \frac{1}{8}\lambda\left(\varphi^2 - v^2\right)^2 \tag{4.1}$$

with $\varphi^2 = \varphi^i\varphi^i$ and $i = 1, 2, 3$. Clearly the potential can be minimized for $\varphi^i = v(\varphi^i/|\varphi|)$, where $|\varphi|$ is just $\sqrt{\varphi^i\varphi^i}$. This implies that the boundary condition on our scalar fields should be something like $\varphi^i(r = \infty) \sim v\hat{x}^i$, which is just the three dimensional analogue of what we did in section 3. Furthermore, we know that we will be unable to find finite energy solutions unless we throw some gauge fields into the mix. One compatible model places the scalar fields in the adjoint representation of an $SU(2)$ gauge group, so $\varphi = \frac{1}{2}\varphi^a\sigma^a$ where the σ^a are the familiar Pauli matrices. This theory is described by the Lagrangian

$$\mathcal{L} = -\frac{1}{2}(D^\mu\varphi)^a(D_\mu\varphi)^a - V(\varphi) - \frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} \tag{4.2}$$

with $V(\varphi)$ given by (4.1), and the covariant derivative in the adjoint representation given by $(D_\mu\varphi)^a = \partial_\mu\varphi^a + e\epsilon^{abc}A^b_\mu\varphi^c$. We can understand the transformation properties of the scalar field on two levels. Trivially, we noticed that we would achieve a moduli space (the space of vacuum configurations) that was topologically the same as our spatial boundary by studying three real scalar fields in the potential (4.1). The adjoint representation of $SU(2)$ is $2^2 - 1 = 3$ dimensional, and thus a perfect fit for our scalars (of course they don't *have* to transform as an irreducible representation, but it makes life easier).

More importantly, if we decide to break the gauge symmetry along, say, φ^3 (we can always do this, as all vacuua in the moduli space are related by a global $SU(2)$ rotation) then the matrix valued VEV ($\mathbb{V}$) is proportional to σ^3: $\mathbb{V} = \frac{1}{2}v^a\sigma^a = \frac{1}{2}v\sigma^3$. Obviously, $\mathbb{V}$ commutes with σ^3, and thus this generator forms an unbroken $U(1)$ subgroup. All of this is a fancy way of saying that by putting the scalars in the adjoint representation, we can break the gauge symmetry in such a way as to leave one of the three vector bosons (A^3_μ) massless. This massless gauge boson is associated with an unbroken $U(1)$ gauge symmetry, and thus we will have a certified photon. This should be contrasted with the case of the vortex, where the "photon" acquires a mass when the gauge symmetry breaks.

For the time being, we will focus our attention on the special case of the identity map from $S^2_\infty \mapsto S^2_v$. This corresponds to the boundary condition on the scalar field

$$\varphi^a(x) \to v\frac{x^a}{\sqrt{x^a x^a}} = v\frac{x^a}{r} \tag{4.3}$$

at the boundary of space. Just as before, we can use the gauge field to keep the kinetic contribution to the excitation energy finite. This amounts to requiring $(D_\mu\varphi)^a = 0$ as we approach $r \to \infty$. Using the definition of the adjoint covariant derivative, together with (4.3), it is simple to show that the asymptotic form of the gauge field must be

$$A^a_i(x) \to \epsilon^{aij}\frac{x^j}{er^2} \tag{4.4}$$

and thus our trial solutions will take the form $f(r)$ times (4.3) and $g(r)$ times (4.4), where f and g are functions we will determine by solving the equations of motion (just like before).

Using the nonabelian generalization of (3.10), it is easy to see that the energy we should be minimizing is given by

$$E = \int d^3x \left[\frac{1}{2}(D_i\varphi)^a(D^i\varphi)^a + V(\varphi) + \frac{1}{2}B^a_i B^a_i\right] \tag{4.5}$$

where, as one might have guessed, the Yang-Mills magnetic field is given by $B^a_i = \frac{1}{2}\epsilon_{ijk}F^a_{jk}$. At this juncture, we have several options. By requiring $\delta E = 0$, we could obtain the "equations of motion" for our system, and subsequently insert our ansatz for φ and A_μ. This will result in a system of differential equations for $f(r)$ and $g(r)$ which we can then try to solve, constrained by the appropriate boundary conditions. A considerably easier (and equally enlightening) course of action is to see if we can write a Bogolmol'nyi inequality, and construct the BPS states that saturate it. We will eagerly opt for this approach.

The goal is to write (4.5) as a constant plus an integral over manifestly positive terms. The constant will come from a surface term in the integrand, so we should look for a clever way to redistribute derivatives. To this end, note that

$$\frac{1}{2}(D_i\varphi)^a(D^i\varphi)^a + \frac{1}{2}B_i^aB_i^a = \frac{1}{2}\left[B_i^a + (D_i\varphi)^a\right]^2 - B_i^a(D_i\varphi)^a \tag{4.6}$$

The first term is certainly always positive, and the second looks as though it could house our sought-after surface term. To find the desired term, we will need to use a few facts from differential geometry. First, covariant derivatives satisfy a Leibniz rule: $D(P \otimes Q) = (DP) \otimes Q + P \otimes (DQ)$. We can put this in the gauge theory context by remembering that a product of fields transforms in the direct product representation. Since B and φ both transform in the adjoint, we can use the $SU(2)$ fact $3 \otimes 3 = 1 \oplus 3 \oplus 5$ to note that the product $B\varphi$ can transform as a singlet. Of course the covariant derivative of a singlet field is just the ordinary partial derivative, so we find that

$$D_i(\varphi^aB_i^a) \to \partial_i(\varphi^aB_i^a) = (D_i\varphi)^aB_i^a + \varphi^a(D_iB_i)^a \tag{4.7}$$

This is promising, as the "leftover" term in (4.6) is evidently equal to the difference of a surface term and a divergence of B. Evidently, we would be in good shape if this divergence vanished. Here we appeal to a Bianchi identity, which is guaranteed for covariant derivatives of fields in the adjoint representation by virtue of the Jacobi identity. Explicitly,

$$D_iB_i = \frac{1}{2}\epsilon^{ijk}D_iF_{jk} = \frac{1}{2}D_{[i}F_{jk]} = 0 \tag{4.8}$$

where the Bianchi identity is used in the last equality. Finally, we are able to rewrite our energy integral in a useful way:

$$E = \int d^3x \left[\frac{1}{2}\left[B_i^a + (D_i\varphi)^a\right]^2 + V(\varphi) + \right] - \int d\Omega_i B_i^a\varphi^a \tag{4.9}$$

where $d\Omega$ is the surface element of a sphere located at infinity. As promised, we have a certifiable Bogomol'nyi bound

$$M \geq v|Q| \equiv -\int d\Omega_i B_i^a\varphi^a \tag{4.10}$$

and a condition for BPS states (taken for vanishing λ)

$$B_i^a + (D_i\varphi)^a = 0 \tag{4.11}$$

which we will tackle in the order presented.

The integral in (4.10) has a very obvious interpretation, which likely won't come as much of a surprise given the conclusions section 3. On the surface of interest, at spatial infinity, we have $\varphi^a \to v\delta^{a3}$, providing we agree to break the symmetry along the third component of the scalar field. As discussed previously, this is a smart choice because it leaves the gauge boson A_μ^3 massless, giving rise to a wonderfully ordinary electrodynamics.

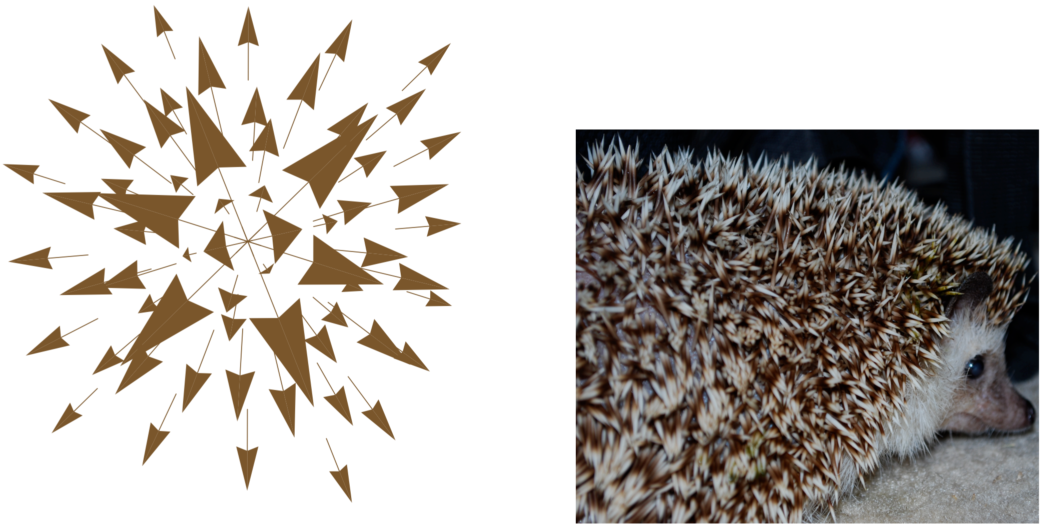

Figure 5: [Left] The BPS 't Hooft-Polyakov monopole (hedgehog). [Right] The monopole's namesake, my deceased hedgehog Leibniz.

The Kronecker delta picks out B^3, leaving us with the integral of an honest-to-goodness magnetic field over a sphere at infinity—the magnetic flux!

$$|Q| = -4\pi \int drr^2 B^3 = \frac{4\pi}{e} \tag{4.12}$$

In computing the integral (4.12), it is important to remember that B^3, computed from the field strength $F^3_{ij} = \partial_i A^3_j - \partial_j A^3_i$ can only be interpreted as the magnetic field at the *boundary* of space. Although we were able to avoid the issue here, B^3 actually transforms like a component of the gauge field under gauge transformations—it is not gauge invariant. This implies that an improved, gauge invariant definition of the electromagnetic field strength should be used everywhere else in space. Such a tensor is easy to construct, see for example [2].

Summarizing, we have found in (4.12) the magnetic flux emanating from a point particle. Our solution is thus a stable particle in a 3+1 dimensional field theory that carries magnetic charge $g = -(4\pi/e)$—in other words a magnetic monopole. The particular defect we have been studying has a special name: it's the 't Hooft-Polyakov monopole. If we were to back up a bit and generalize our map for arbitrary winding number n, it is not hard to show that the flux is again quantized like $g = n(4\pi/e)$. This brings us to one of the most celebrated results of all time. Note that an "electron", sitting at the origin, produces an electric flux $\Phi_E = q/(4\pi r^2) \cdot 4\pi r^2$ equal to *its* charge, q. If we add "electrons" into our theory, placing them into the fundamental representation of $SU(2)$, they will carry charge $q = e/2$. Combining these results, we have

$$qg = 2\pi n \tag{4.13}$$

which is nothing more than Dirac's fabled charge quantization condition. In section 4.1 we will briefly return to this result, deriving it in a far more elegant manner.

It would be a shame if we left this section without first writing down an explicit solution for the 't Hooft-Polyakov monopole. From (4.11), it is straightforward to derive a coupled system of differential equations describing the BPS state with mass gv. These equations are not very impressive, but their exact solutions certainly are:

$$\varphi^a(r) = v\left(\coth(evr) - \frac{1}{evr}\right)\hat{x}^a \tag{4.14}$$

$$A_i^a(r) = \frac{1}{er^2}\left(1 - \frac{evr}{\sinh(evr)}\right)\epsilon^{aij}x^j \tag{4.15}$$

The field configuration described by (4.14) is shown in figure 5. For obvious reasons, the 't Hooft-Polyakov monopole is often called a hedgehog.

4.1 From Another Angle

There remains one final point worth typing, which showcases the underlying geometric nature of the magnetic monopole. Given a two-form field strength, F, we can compute the magnetic flux corresponding to the charge of the monopole by integrating F over S^2 as in (3.6). For better or worse, this isn't always the painless procedure we made it out to be in our study of the vortex. The problem is that $F = dA$ is not in general *globally* exact, and so multiple charts may be needed to properly define the integral. This is exactly what happens for the monopole–since one needs two coordinate patches (N, S) to cover the sphere,

$$g = \int_{S^2} F = \int_N dA_N + \int_S dA_s = \int_{S^1} A_N - \int_{s^1} A_S \tag{4.16}$$

Clearly, if the gauge potential is globally defined, $(A_N = A_S)$ the flux vanishes and we have no monopole. The fact that our vortex solution arose in a two dimensional theory saved us from this worry earlier. The problem is thus reduced to finding two locally defined potentials—one for the northern hemisphere (A_N) and one in the south (A_S)—that give rise to the monopole field strength $F = -(g/4\pi)\sin\theta d\theta d\phi$.

This problem has a well known solution, originally due to Wu and Yang [9]. Choose

$$A_N = \frac{g}{4\pi}(\cos\theta - 1) \tag{4.17}$$

$$A_S = \frac{g}{4\pi}(\cos\theta + 1) \tag{4.18}$$

so that there is no ambiguity at $\theta = 0$ in the north, and $\theta = \pi$ in the south. Now we simply take these potentials and plug them in to (4.16). Since we want both potentials to describe the same physics, we require them to be related by a gauge transformation: $A_N - A_S = (g/2\pi)d\phi = (1/iq)e^{-i\chi}de^{i\chi}$. Accordingly,

$$g = \int_{S^1} A_N - A_s = \frac{1}{iq}\int_{S^1} e^{-i\chi}de^{i\chi} = \frac{1}{e}\int_{S^1}\partial_\mu\chi dx^\mu = \frac{1}{q}\left[\chi(2\pi) - \chi(0)\right] \tag{4.19}$$

Because the gauge function $\chi(x)$ appears as a phase, $\chi(x)$ and $\chi(x) + 2\pi n$ give the same gauge transformation. In other words, $\chi(2\pi) - \chi(0) = 2\pi n$ and so for the second time in as many pages we find

$$qg = 2\pi n \tag{4.20}$$

This approach emphasizes the important connection between quantized charge, magnetic monopoles, and (compact) $U(1)$ gauge symmetries, about which we will have more to say in the conclusion.

5. When All is Said and Done

The observation that field theories are capable of producing non-perturbative excitations has, at this juncture, provided us with far too many pages of entertainment. That said, it will prove useful to fill a final page with a sort of highlight reel to take stock of where we've been…

By searching for time-independent field configurations that minimized the energy, we were able to select candidates for soliton solutions. In all cases, we turned our attention to theories with spontaneously broken symmetry, because non-trivial moduli spaces coupled with the appropriate dimensionality of space guaranteed the stability of our defects. This stability was a consequence of the theory of homotopy groups, which organized for us the set of all topologically inequivalent maps from the spatial boundary to the space of vacuum configurations.

To move beyond d = 1+1, we were required by Derrick's theorem to introduce vector fields into the mix. Through the covariant derivative, these new gauge fields were tailored to keep the kinetic contribution to the rest mass finite. In our d = 2+1 theory, the gauge fields acquired a mass, and our soliton solution spit out quantized "magnetic" flux. This solution whirled about its core, so we called it a vortex. We noted that a massive photon, coupled with vortices capable of transporting magnetic fields, was exactly what one might hope for in an attempt to describe the phenomenology of superconductors.

Incrementing the dimension of space by one brought a wealth of new technicalities to address, as well as some new physics. By studying a theory with spontaneous symmetry breaking scalars in the adjoint representation of an $SU(2)$ gauge symmetry, we were able to isolate an unbroken $U(1)$ theory we called electrodynamics. The stable, time independent minimum energy configurations we found carried quantized magnetic charge, and were thus dubbed magnetic monopoles. As an example of such a defect, we studied the BPS states of the 't Hooft-Polyakov monopole, finding that they very much resembled a small mammal covered in spines. We then tried for a deeper understanding of the Dirac charge quantization condition, looking at the problem through the eyes of a geometer. Here we learned that the existence of a monopole (and hence charge quantization) was fundamentally related to the existence of a compact $U(1)$ gauge group that doesn't appear as a direct product. This is the case for grand unified theories based on simple Lie groups like the Georgi-Glashow $SU(5)$, but *not* for the standard model, where the gauge group is given by $SU(3) \otimes SU(2) \otimes U(1)$. It is likely appropriate to point out that magnetic monopoles have yet to be seen experimentally.

Finally, it may (or may not) be painfully obvious that in all these pages, there is no mention of the quantum theory of solitons, nor their interactions. These are rich extensions equally deserving of ample air time. Interacting quantum mechanical solitons arise in many diverse corners of physics, from the study of condensed matters to supersymmetric string theories. In fact, these applications highlight one of the most remarkable features of these topological defects, which is their role in dual models of physical systems. That, however, is another story.

References

[1] M. Srednicki, *Quantum Field Theory*; Cambridge (2007)

[2] A. Zee, *Quantum Field Theory in a Nutshell*; Princeton (2003)

[3] S. Coleman, *Aspects of Symmetry*; Cambridge (1995)

[4] E. Kiristas, *String Theory in a Nutshell*; Princeton (2007)

[5] M. Nakahara, *Geometry, Topology and Physics*; (2003)

[6] H.B Nielsen and P. Olesen, *Nuclear Physics* B61 (1973)

[7] L. Susskind, *Nuovo Cimento* 69A (1970)

[8] The author and his collaborators hope this is sooner rather than later...

[9] T.T. Wu and C.N Yang, *Phys. Rev. D* 12 (1975)

[10] J. Harvey, *Magnetic Monopoles, Duality, and Supersymmetry* arxiv:hep-th/9603086