Parting the Partons

Chris Rosen

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At energy densities greater than a few GeV/fm^3, lattice QCD calculations predict the formation of a new state of matter. Above this critical temperature ($T_c \approx 175$ MeV [1]), partons are deconfined from one another and exist as a quark-gluon plasma. Here I examine an ideal gas of quarks and gluons in the relativistic limit. I start from the full relativistic dispersion relation, and determine a number of interesting statistical quantities such as the parton number and energy densities as functions of the plasma temperature.

1 A Physics Introduction

An interesting question to ask in physics is whether or not the atoms and particles of today arrange themselves in the same ways as atoms and particles did at the beginning of the universe. In an attempt to answer this rather daunting question, one can step backward through our universe's evolution and examine the conditions that existed in it's youth.

About 400,000 years after our universe popped onto the scene by way of the big bang, previously free electrons began to notice the positively charged nuclei that surrounded them in space. Before this time, the electrons had sufficient energy to resist the electromagnetic interaction, and formed a dense scattering trap which effectively corralled the photons. As soon as the electrons could no longer escape the tug of the Coulomb interaction, they bound to the nuclei setting the trapped photons free in what would later be called the cosmic microwave background radiation. This process is known as recombination.

Somewhere around 5 seconds after the big bang, the speeding protons and neutrons begin to feel the effects of the strong force. This is the age of nucleosynthesis. Before this time, the universe was sufficiently hot for the protons and neutrons to roam about indifferent to one another. Since unbound neutrons have a half life of something like 15 minutes [2], some of them even decayed. At some point, seconds after it all began, the nucleons had enough energy to bring them close enough to one another for the strong force to bind them, but not enough to get away. These nuclei would later go on to form the atoms in our universe today.

Mere microseconds after the beginning, it gets more interesting yet. The universe in this era is now over a trillion degrees—hot enough for the quarks and gluons (that would later confine themselves into the nucleons) to exist in a deconfined soup known as the quark-gluon plasma. This plasma represents a state of matter that hasn't existed for over 14 billion years, and it's observation represents one of the most formidable challenges in the physics of heavy ion collisions.

Quantum chromodynamics (QCD), the prevailing theory of the strong interactions, predicts that a quark-gluon plasma (QGP) can exist at extremely high temperatures and energy densities—somewhere around 5 GeV/fm^3 [1]. Experimentally recreating these conditions is one of the primary research goals of the Relativistic Heavy Ion Collider (RHIC) at Brookhaven National Labs. At RHIC, heavy ions such as gold nuclei (favored for their sizable mass and spherical shape) are accelerated to nearly 99.99% the speed of light and collided into one another at a collision vertex surrounded by a myriad of detectors. These collisions put a lot of energy (up to 40 TeV) into a minuscule space, and may have a shot at observing a QGP.

In this work, I explore some of the consequences of a relativistic soup of quarks and gluons. I begin by considering the case that the temperature far exceeds the rest mass of the quark (the high temperature limit) and show that there are simple, analytic results for the ratios of the number and energy densities between quarks and gluons. I then investigate numerically how these ratios change as the temperature sinks below the quarks rest mass. Ultimately, these methods are generalized and discussed as to whether or not they might provide a rough portrait of the produced matter within a heavy ion collision.

2 The High Temperature Limit

As a massless gauge boson, the gluon travels at the speed of light and is described with the familiar dispersion relation

$$\epsilon_k = \sqrt{(\hbar c k)^2 + (mc^2)^2} \tag{1}$$

where $m = 0$. In general, the density of states, $g(\epsilon)$, is given by

$$g(\epsilon) = \frac{g_s}{(2\pi)^3} \int d\vec{k}\, \delta(\epsilon - \epsilon_k) \tag{2}$$

where g_s is the degeneracy factor. Gluons themselves have color charge, in fact each carries one unit of color and one unit of anti-color. Since there are three varieties of color, one would expect 9 distinct gluons. As it turns out, one is a color singlet [3] and accordingly the contribution to the gluon degeneracy from the color is eight. The gluon degeneracy factor must also account for two possible spin states, so we get a grand total of $g_s = 8 \times 2 = 16$.

Evaluating this integral is best accomplished by exploiting the (spherical) symmetry of the problem, and noting that the k dependence of the delta function requires multiplication by a Jacobian term:

$$\begin{aligned} g_g(\epsilon) &= g_s \frac{4\pi}{(2\pi)^3} \int k^2\, \delta(\epsilon - \epsilon_k) dk & (3) \\ &= g_s \frac{4\pi}{(2\pi)^3} k^2 \frac{1}{\left|\frac{\partial \epsilon_k}{\partial k}\right|}\Bigg|_{k=\frac{\epsilon}{\hbar c}} & (4) \\ &= \frac{8\epsilon^2}{\pi^2(\hbar c)^3} & (5) \end{aligned}$$

A similar analysis can be preformed for the case of quarks and anti-quarks in the limit where $k_B T >> mc^2$. Here the dispersion relation is identical to that of the gluon's with a modified degeneracy term. Since the quarks come in three colors and two spin states, the appropriate degeneracy factor is $g_s = 6$. Integrating just as we did for the gluons, we find that

$$g_q(\epsilon) = \frac{3\epsilon^2}{\pi^2(\hbar c)^3} \tag{6}$$

gives the density of states for the quarks.

We can exploit the grand potential to determine the number and energy densities of these hot quarks and gluons by noting that

$$\Pi = k_B T \ln \Xi = pV \tag{7}$$

where Ξ is the grand partition function. Under the quantum mechanical representation, Ξ can be written in a surprisingly useful form if one agrees to represent the system's energies as the discrete eigenvalues of some Hamiltonian:

$$\mathscr{H}|i\rangle = E_i|i\rangle$$

The conservation of baryon number, b, guarantees that the net number of baryons never changes under any physics interaction, so it is a good quantum number in this analysis. Like any good quantum number, the operator associated with it ($\hat{b}$ in this case) commutes with the Hamiltonian, which is one way of saying that the energy eigenstates can also be characterized by their baryon number. With this, one can return to the grand-canonical partition function and attempt to write it in a more revealing way. If $\mathcal{Z}$ is the canonical partition function, then

$$\begin{aligned} \Xi(T, V, \mu) &= \sum_i e^{\beta\mu b_i} \mathcal{Z} \\ &= \sum_i e^{-\beta(E_i - \mu b_i)} \\ &= \sum_{i,b} \langle i, b| e^{-\beta(\mathscr{H} - \mu\hat{b})} |i, b\rangle & (8) \end{aligned}$$

Equation (8) is extraordinarily useful, because it simply represents the trace of an operator related to the grand-canonical partition function. Since the trace of any quantum mechanical operator is representation independent [4], one can choose the occupation number basis $|n\rangle$ to expand the states.

In the occupation number basis, the state energy is simply the sum of all the level energies multiplied by how many particles are in each level, or

$$E_n = \sum_i n_i \epsilon_i$$

Now, evaluating the trace in the occupation number basis gives the grand-canonical partition function as

$$\Xi = \sum_n e^{-\sum_i n_i \beta(\epsilon_i - \mu b_i)}$$

This expression can be simplified greatly, especially if one is working towards the grand potential (7). Fortunately, this is exactly what we are in the process of doing—the answer is

$$p_{f/B}(T,\mu) = \pm \frac{k_B T}{V} \sum_i \ln(1 \pm e^{-\beta(\epsilon_i - \mu b_i)}) \quad (9)$$

where the plus sign is for fermions and the minus for bosons. The summation in (9) can be considered an integral over level energy as long as one multiplies the integrand by the density of states. In this case

$$p_{f/b}(T,\mu) = \pm k_B T \int g(\epsilon) \ln(1 \pm e^{-\beta(\epsilon - \mu b)}) d\epsilon \quad (10)$$

As a matter of notation, it is handy to note that the eigenvalue of $\hat{b}$ is the opposite sign for particles and anti-particles. Instead of carrying around the b for the next few pages, one can simply introduce this sign change into the chemical potential, so that

$$\mu = -\bar{\mu} \quad (11)$$

where $\bar{\mu}$ denotes the chemical potential of an anti-particle.

When there are both quarks and anti-quarks present, the grand potential contains grand partition function contributions from both the quarks and the anti-quarks, and it follows that the fermion pressure should as well:

$$\begin{aligned} p_{q,\bar{q}}(T,\mu) &= k_B T \int g(\epsilon)[\ln(1 + e^{-\beta(\epsilon-\mu)}) \\ &\quad + \ln(1 + e^{-\beta(\epsilon - \bar{\mu})})] d\epsilon \end{aligned} \quad (12)$$

From (10), the number density is readily calculated as the derivative of the pressure with respect to μ holding the temperature fixed. Differentiating in this way yields

$$n(T,\mu) = \int \frac{g(\epsilon)}{e^{\beta(\epsilon-\mu)} \pm 1} d\epsilon \quad (13)$$

as the number density for quarks or anti-quarks (plus sign), or gluons (minus, $\mu = 0$).

If there are equal numbers of quarks and anti-quarks, it follows that $\Delta N \equiv n_q - n_{\bar{q}} = 0$, and something very interesting happens. Subtracting the number of anti-quarks per unit volume from the number of quarks per unit volume (as given by (13)) gives zero so long as $\mu = \bar{\mu}$. From (11), one can see that this requires $\mu = -\mu$ which is clearly only true if $\mu = \bar{\mu} = 0$.

When $\mu = 0$, the number density is easily integrated. Under the substitution $x = \beta\epsilon$ the integral in (13) is of the form

$$\int \frac{x^2}{e^x \pm 1} dx \quad (14)$$

and can be evaluated in terms of polylogs (q's, $\bar{q}$'s) and Riemann-zeta functions (gluons). As it turns out, so long as $\mu = 0$ the polylogs themselves can be written in terms of the zeta functions, and the analytic number densities in the high temperature limit are simply

$$n_{q,\bar{q}}(T,\mu=0) = \zeta(3) \frac{9(k_B T)^3}{\pi^2 (\hbar c)^3} \quad (15)$$

for the quarks and anti-quarks and

$$n_g(T) = \zeta(3) \frac{16(k_B T)^3}{\pi^2 (\hbar c)^3} \quad (16)$$

for the gluons. The most interesting quantity in this work will be the ratio of the quark-gluon densities. In this (hot) limit, one finds

$$\frac{n_q}{n_g} = \frac{9}{32} \quad (17)$$

so at temperatures far greater than the rest mass of the quark, there are roughly 4 gluons for every quark.

It's not too much harder to find the quark and gluon energy densities. The energy density is simply the number density weighted by the parton energy, or

$$u(T, \mu = 0) = \int \epsilon \, g(\epsilon) f(\epsilon) d\epsilon \tag{18}$$

where $f(\epsilon)$ is the Bose or fermion occupancy factor. Just as before, these integrals can be evaluated explicitly with the help of the zeta functions, they give

$$u_{q,\bar{q}} = \zeta(4) \frac{63(k_B T)^4}{\pi^2 (\hbar c)^3} \tag{19}$$

for the quarks and anti-quarks, and

$$u_g = \zeta(4) \frac{48(k_B T)^4}{\pi^2 (\hbar c)^3} \tag{20}$$

for the gluons. The ratio of these two densities shows that they are directly proportional to each other, namely

$$\frac{u_q}{u_g} = \frac{21}{64} \tag{21}$$

so the gluons have about three times as much energy per volume as the quarks.

Since the energy per parton is simply the parton's energy density divided by its number density, it is a trivial matter to arrive at the average energy per quark (or anti-quark). Dividing (19) by (15) leaves

$$U_{q,\bar{q}} = (k_B T) \frac{63\zeta(4)}{18\zeta(3)} \tag{22}$$

This energy is particularly interesting because it is easily related to the volume described by the wavelength of an averagely energetic quark. In the high temperature limit the quarks are all traveling at approximately the speed of light, so they have a wavelength determined by the deBroglie relation

$$\lambda = \frac{hc}{U} \tag{23}$$

$$= hc \frac{\zeta(3) 18}{(k_B T) \zeta(4) 63} \tag{24}$$

The number of quarks in a volume λ^3 is just the number density of quarks multiplied by (23), or

$$n \cdot \lambda^3 = 36\pi\zeta(3) \left(\frac{18\zeta(3)}{63\zeta(4)} \right)^3 \tag{25}$$

This expression can be evaluated numerically, and it says that there are about 4 quarks in each volume described by the quark's wavelength. Of course the same is true of the anti-quarks.

3 The Not-So-Hot Limit

As the temperature cools, it is no longer appropriate to ignore the quark mass term in equation (1), and the physics gets far more interesting. Specifically, it is in this limit that one must begin to pay careful attention to the physicality of the statistical model. In the final section of this work, I will spend some time discussing how a QGP may (or may not) resemble a relativistic partonic gas. At present, it will suffice to note that if it is *ever* appropriate to model a QGP with such a gas, it *only* makes sense to do so when the temperature exceeds about 175 MeV, the critical temperature that marks the phase transition from ordinary hadronic matter to the plasma. In this section, I derive the quark to gluon number and energy density ratios as functions of the temperature, and plot them as such. It is important to keep in mind that below $T_c \approx 175$ MeV hadronization has occurred and the pictures are largely meaningless. I have chosen to include the plots in their entirety because their shapes are particularly interesting—while I consider systems consisting of the three lightest quark flavors (figures 3,4), with some imagination one can see that, at higher temperatures, similar curves will describe systems consisting of the heavier quarks as well.

In this limit, the dispersion relation can be integrated to yield the modified density of states

$$g(\epsilon)_q = \frac{3}{\pi^2 (\hbar c)^3} \epsilon \sqrt{\epsilon^2 - (mc^2)^2} \tag{26}$$

which clearly approaches (6) as $m \to 0$. This density of states describes some interesting features, as can be seen in figure 1. Primarily, there are no states below the rest energies of the quarks, and the densities of states approach each other as the energy becomes very large. More specifically, at energies near the rest energy the density of states grows like the square root, then like the energy squared as ϵ increases.

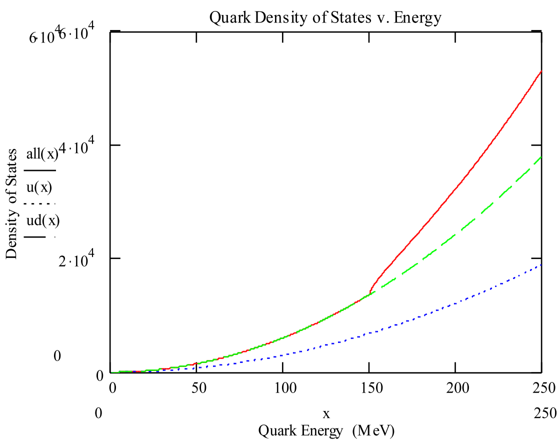

Figure 1: Plot of the density of states against energy for the three lightest quarks. The blue curve represents the density of states for the up quark alone, the green curve gives the density of states for a system of up and down quarks, and the red curve shows the density of states for a system consisting of up, down, and strange quarks. Notice that in the latter case, there is no contribution to the density of states from the strange quark until the energy is above its rest energy (near 150 MeV).

The quark and anti-quark number densities are again given by (13), this time however the integral has no closed form analytic solution. Compared to the number density of the gluons (16),

$$\frac{n_q}{n_g} = \frac{3}{\zeta(3)16}\int \frac{x\sqrt{x^2-(\beta mc^2)^2}}{e^x+1}dx \qquad (27)$$

where the integral has been made dimensionless through the substitution $x = \beta\epsilon$. From this expression, one can note that at high temperatures (as $\beta mc^2 \to 0$), one ends up with exactly the same result given by (17).

When the temperature is very small (large β), the exponential in the integrand of (27) sends the ratio quickly towards zero, like $\exp(-\beta mc^2)$, reflecting the fact that at lower temperatures there are many more gluons than quarks. A better description of this ratio can be found in the plots of figure 2. This graph was made by numerically integrating the expression in (27) for many values of βmc^2. As expected, in the

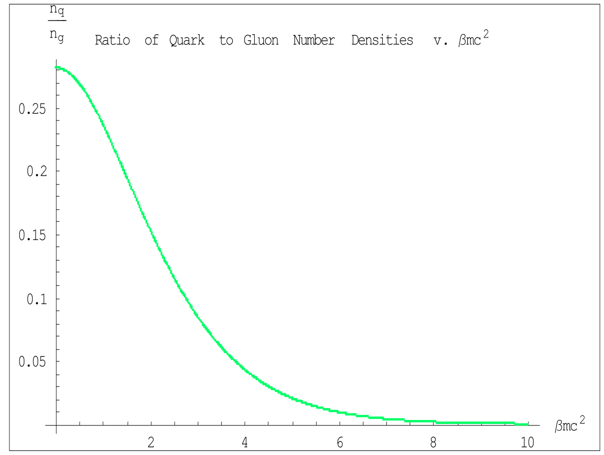

Figure 2: Generic Plot showing the ratio of the number of quarks to the number of gluons for a range of values of βmc^2.

high temperature regime the ratio is constant and around $n_q/n_g = 0.281$ which is precisely the ratio predicted by (17). As the temperature decreases the ratio drops sharply as well. By the time the temperature has decreased to 1/5 the quark's rest mass, the ratio has already fallen to about 1 quark for every 50 gluons.

An equally interesting plot is shown in figure 3, which shows a log-log plot of the ratio of the number of quarks to the number of gluons for the three lightest quark flavors as a function of β. Just as before, at high temperatures there is a contribution of about 0.281 from each of the three flavors. As the temperature declines, the ratio is seen to fall abruptly with a "step", marking the abscence of the strange quark. No step is seen when the system transitions from up and down quarks to up quarks alone because the two flavor's rest energies are very similar.

Not surprisingly, it is fairly simple to determine the energy density ratio in this section's (not-so-hot) limit. With the appropriate dispersion relation, one finds

$$\frac{u_q}{u_g} = \frac{3}{\zeta(4)48}\int \frac{x^2\sqrt{x^2-(\beta mc^2)^2}}{e^x+1}dx \qquad (28)$$

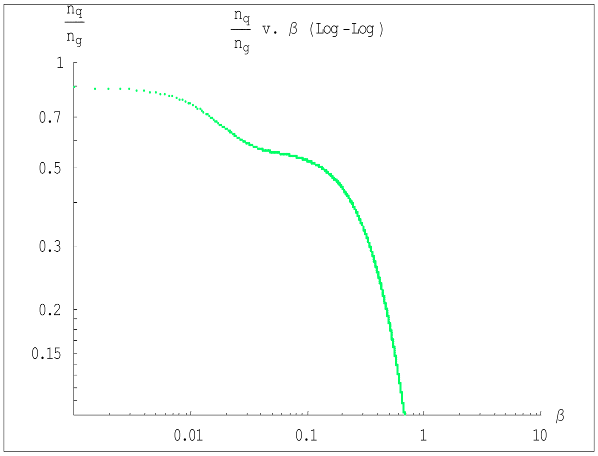

Figure 3: Log-log Plot showing the ratio of the number of quarks to the number of gluons for a system of up, down, and strange quarks. Ratio is plotted for a range of temperatures. The first "step" shown at high temperatures (small β) reflects the influence of the disappearance of the strange quark.

where the same substitution ($x = \beta\epsilon$) has been used to rid the integral of dimension. This ratio is plotted in figure 4 for a system containing the three lightest quarks. Conveniently, the limit of (28) as $\beta mc^2 \rightarrow 0$ exactly approaches that obtained in (21), about 0.328. As the temperature decreases, the exponential in the denominator again dominates and the ratio shoots toward zero, reflecting the fact that at low temperatures the energy density of the gluons far surpasses that of the quarks.

4 From the Very Big to the Very Small

Although the statistical description of the partonic gas seems to offer a very nice glimpse of some properties of a quark-gluon plasma, it is fair to wonder how well a plasma filled with quarks and gluons can truly be described with these methods. To answer this question, one can begin by discussing the assumptions that were used to develop the statistical model.

One interesting assumption made throughout this work was that there is no baryon/anti-baryon asymmetry in the QGP. This assumption required that the quark and anti-quark chemical potentials were both zero, and greatly simplified the integration for the number and energy densities. In fact, heavy ion collisions have an inherent matter/anti-matter assymmetry, as two colliding gold nuclei bring with them $2\times(79\times3+118\times3) = 1182$ quarks and no anti-quarks. Since baryon number is conserved, this asymmetry will always exist in the collision: no matter how many particles and anti-particles are produced in the fireball, at the end of the day there will always be 1182 more quarks than anti-quarks. There is no analagous 'conservation of boson number' for the gluons.

The 1182 quarks that enter the collision are the valence quarks of the nucleons in the heavy ions. Since both protons and neutrons are comprised of up and down quarks alone, no additional quark flavors are brought into the collision. This observation, coupled with the fact that the rest mass of the strange quark is relatively close to T_c, suggests that the study of strange particles produced in heavy ion collisions might be of particular interest.

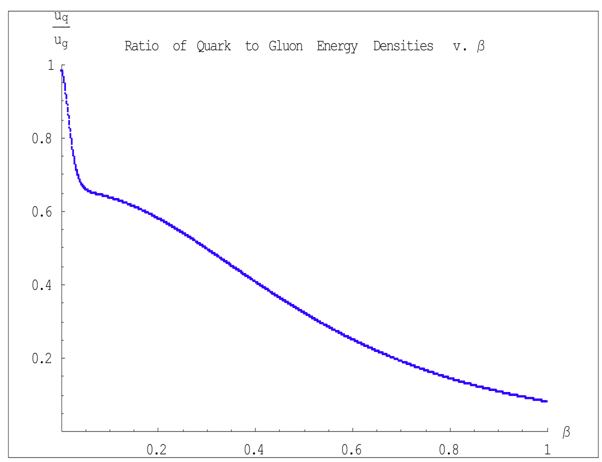

Figure 4: The above plot shows the ratio of the energy of a system of up, down, and strange quarks to the energy of the gluons for many values of β. Note the steep decline in the energy density of the quarks as it becomes too cold to make more strange quarks.

In the quark-gluon plasma, the primary means of producing strange particles is through a process called gluonic fusion, in which $gg \rightarrow s\bar{s}$ [4]. From figure 2, one can see that these produced strange and anti-strange quarks decline noticeably in number by the time the critical temperature is reached ($\beta mc^2 \approx 1$) and hadronization occurs. Because the masses of the up and down quarks are small compared to T_c, the decrease in their number is far smaller. With this, one might suspect that if the gluonic fusion process contained some inherent matter/anti-matter asymmetry, it might be reflected in the final state strange particles that emerge from the fireball.

As an exercise, I arbitrarily chose a strange/anti-strange asymmetry of 1 and plotted the number densities as a function of temperature in figure 5. From

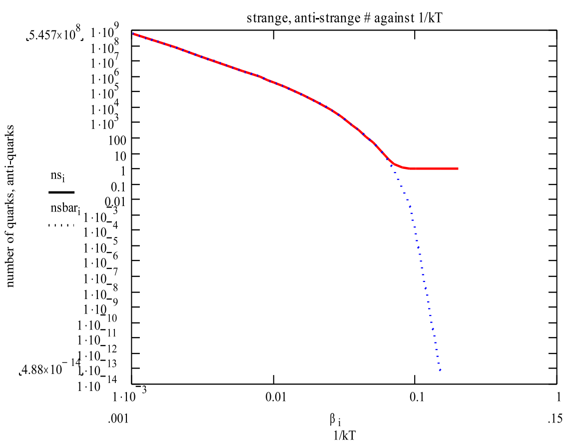

Figure 5: Log-Log plot showing the number of strange (red line) and anti-strange (blue dots) quarks as a function of β for non-zero baryon number. In this case the strange/anti-strange asymmetry has been set to one. At temperatures less than about 10 MeV there are essentially no anti-strange quarks, and one strange quark remaining.

this picture, one can see that at or before the temperature has fallen to the critical temperature there are on the order 10^7 strange quarks, and $10^7 - 1$ anti-strange quarks. This result asks the interesting experimental question of how well strange particles can be counted. Evidently, if the matter/anti-matter asymmetry in gluonic fusion was much, much larger, it might be possible to see this asymmetry more clearly in the multitudes of strange particles formed during hadronization. Studies of charged kaon ratios suggest that it is not [5].

Perhaps more important yet is the assumption that the individual partons interact only weakly with their neighbors. There are two fundamental forces that one could be worried about. First, since the quarks and anti-quarks carry electric charge (gluons are electrically neutral), one must consider the possibility of Coulomb interaction. Fortunately, at the energies of interest here (above 175 MeV) the cross section for the Coulomb interaction is essentially non-existent. One can show this by dividing the energy density of the Coulomb interaction

$$\begin{aligned} u_C &= n_q \frac{q^2}{4\pi\epsilon_0 \langle d \rangle} \\ &= n_q^{4/3} \frac{q^2}{4\pi\epsilon_0} \end{aligned} \tag{29}$$

by the energy density in the relativistic gas (18), and noting that the quotient is on the order of the fine structure constant ($\alpha \approx 1/137$), far smaller than unity.

The other interaction that may occur within the plasma is the strong interaction between the quarks and gluons. As the name conveniently reminds us, the coupling constant (α_s) for the strong interaction far exceeds the fine structure constant at familiar temperatures. In fact, it is on the order of one, which renders many perturbative calculation methods useless for understanding quark interactions on most energy scales [3]. But all hope is not lost. Curiously, the strong coupling constant *decreases* as the energy increases, a result that earned David Gross, David Politzer, and Frank Wilczek the Nobel Prize in 2004 [6].

The "running" coupling constant's dependence on energy can of course be reflected in temperature as well. So long as the temperature does not exceed about 5 times T_c, a good approximation for $\alpha_s(T)$ is [4]

$$\alpha_s(T) \simeq \frac{\alpha_s(T_c)}{1 + (0.760)\ln(T/T_c)}$$

where $\alpha_s(T_c)$ is calculated to be around 0.5. From this result, one can see that at temperatures at or near the critical temperature, the value of α_s is still not much smaller than one.

From a value for the strong coupling constant at RHIC temperatures, it has been suggested that the produced matter actually behaves more like a fluid than a gas [7]. Indeed, many now think that the partons in heavy ion collisions are strongly interacting, an inference supported by observations of collective phenomena such as elliptic flow [8]. This view of the quark gluon plasma is in opposition to the model I have used, and should instead be treated with the mathematical methods of hydrodynamics. Studies involving such hydrodynamical models have led to a variety of interesting predictions about the QGP, such as it's ability to support collective modes [9].

5 Conclusion

Although the statistical treatment of a relativistic gas of partons may be inadequate to describe the properties of a QGP at RHIC energies, It is unclear whether or not the quarks and gluons will remain strongly coupled as the energy continues to increase [8]. If it is ever appropriate to call a QGP a relativistic gas, then the methods I employed here offer some very nice pictures of the number and energy densities of the plasma constituents. Understanding how these densities change with temperature offers a portrait of a gaseous QGP, and may offer some information about the nature of the QCD interaction. Whether or not that information is of any use is yet to be decided.

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