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All About SU(5): Unification and the Standard Model of Particle Physics

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Abstract: The Standard Model of particle physics, with matter fields transforming under representations of the direct product group $SU(3) \otimes SU(2) \otimes U(1)$, is one of the greatest triumphs of theoretical physics. While highly predictive, this model is also highly tuned. In order to extract meaningful results, it is currently necessary to manually insert about twenty paramters that quantify particle masses and couplings. Motivated by both physics and aesthetics, it is interesting to ask whether there might exist a more cohesive framework on which to build particle physics. Examples of such frameworks are found in the regally named Grand Unified Theories (GUTs) in which matter fields are tucked into representations of a single, larger gauge group (and thus "unified"). Here we explore one such unification scheme, based on the special unitary group SU(5). We begin by highlighting some useful facts about the standard model, and quickly discuss a few group theoretical preliminaries. The remainder of the review is then devoted to embedding the standard model in SU(5) and understanding the consequences of this decision…

Contents

1. Welcome to the Standard Model

Our present understanding of the interactions of subatomic particles represents one of the most detailed descriptions of the natural world ever concocted. The ingredients are a collection of fields whose excitations carry charges and mass. These excitations (read particles) are fundamental, in the sense that they are not composite objects. Interactions between the particles are mediated by gauge bosons. Colloquially, these are the messengers of the strong, weak, and electromagnetic forces. Excluding gravity, all known interactions are accounted for by this model. More importantly, this theory uses these particles and their interactions to produce exceedingly accurate predictions. The calculated muon's anomalous magnetic moment, for example, currently agrees with experiment to something like one part in a million [1].

Interestingly, but perhaps not surprisingly, the Standard Model is deeply rooted in a soil of symmetry principles. Understanding and exploiting these symmetries has been a dominant thrust of particle physics research for more than half a century. This exploration is largely formulated in the language of group theory, and has yielded profound insights into the physics of elementary particles. Early efforts like [2, 3] used approximate symmetries of the strong force to probe this notoriously non-perturbative interaction at energies below Λ_{QCD}. Through clever computation of scattering amplitudes and their ratios, empirical formulas for hadron masses and interaction cross sections were worked out long before the details of the Standard Model (or the strong interaction!) solidified.

While symmetry groups have proved invaluable in formulating our current model of particle physics, the concept of symmetry *breaking* may be more important yet. This far reaching concept allows for an extraordinarily rich phenomenology, and will play a notable role in the second half of this review.

In all its successes, it is not unfair to write that the Standard Model is somewhat utilitarian. Although it provides a framework in which one can calculate to astonishing precision, the fact remains that in order to do so, one must specify about twenty parameters currently dialed by experiment. Is there not some theory in which we need only specify 10, or 2 or none? Furthermore, after the groundbreaking work of Glashow, Weinberg, and Salam, we understand the electromagnetic and weak force as different manifestations of a single "unified" electroweak interaction. Might it be possible to find a theory that adds the strong force into this mix? These are some of the goals of unification. Before we take on these worthy questions, we digress slightly to take stock of what it is we have to work with.

1.1 Gauge Theory Detour

All known interactions between fundamental particles are constrained by the principle of local gauge invariance. In the Standard Model, theories invariant under such spacetime dependent transformations are dubbed gauge theories, and appear in two related but distinct varieties: abelian, and nonabelian. To better understand how these theories give structure to the Standard Model, it will prove beneficial to review some well known facts about field theories in a language convenient for our present purposes.

Importantly, one should recognize that the field operators, $\phi(x)$, in a field theory can be tensor operators that transform in the usual way under the action of a group with generators T^a. To see this, construct a unitary operator called U_T that implements a group transformation on the Hilbert space:

$$\langle f|\phi_i|i\rangle \to \langle f|U_T^\dagger\,\phi_i\,U_T|i\rangle = U^R_{ij}\,\langle f|\,\phi_j\,|i\rangle \tag{1.1}$$

where R labels the representation under which the matrix element transforms, and we sum over repeated indices. Since equation (1.1) holds for arbitrary states $|f\rangle$ and $|i\rangle$, it remains true as an operator equation as well. If we take for concreteness $U_T = \exp(-ig\,\theta_a T^a)$, then for infinitesimal θ_a it is easy to see that (1.1) implies

$$[\,T^a,\,\phi_i\,] = (T^a)^R_{ij}\,\phi_j \tag{1.2}$$

which is precisely the definition of a tensor operator.

To make contact with the standard model, we assume that any valid Lagrangian describing the physics of fundamental particles is invariant under *spacetime dependent* group transformations, $U_T(x)$. Imposing this constraint has substantive implications for the resulting theory. Consider, for example, terms in the Lagrangian like

$$\phi_i^\dagger\phi_i \to \phi_j^\dagger U^{R\,\dagger}(x)_{ji}U^R(x)_{ik}\phi_k \tag{1.3}$$

If the $U^R(x)$ are a unitary representation so that $U^{R\,\dagger}(x)_{ji}U^R(x)_{ik} = \delta_{jk}$, it is obvious that such terms are left invariant. An interesting thing happens, however, when we consider the kinetic term. For the kinetic term to be invariant, we would like the derivative of the field to transform like a tensor operator as well, ie. $D_\mu \phi_i \to U^R(x)_{ij} D_\mu \phi_j$. It is easy to see that the ordinary derivative does not respect this transformation: $\partial_\mu \phi \to \partial_\mu [U^R(x)\phi] \neq U^R(x)\partial_\mu \phi$ as a result of the chain rule. To cancel the unwanted term, we introduce a gauge field, A_μ and build from it the covariant derivative, $D_\mu = \partial_\mu - igA_\mu(x)$. Now for the kinetic term in the Lagrangian to be gauge invariant, it is clear that the derivative must transform in a special way:

$$D_\mu \phi_i \to U_T(x)\ [D_\mu, \phi_i]\, U_T^\dagger(x) = U^R(x)_{ij} D_\mu \phi_j$$

if

$$D_\mu \to U^R(x) D_\mu U^{R\dagger}(x) \tag{1.4}$$

where the derivative in D_μ acts on everything to its right. This transformation law in turn requires the gauge field to transform like

$$A_\mu \to U^R(x)\, A_\mu\, U^{R\dagger}(x) + \frac{i}{g} U^R(x) \left[\partial_\mu\, U^{R\dagger}(x)\right] \tag{1.5}$$

For a theory with an $SU(N)$ gauge symmetry, if the matter fields are in the fundamental representation the $U^R = \exp(-ig\,\theta_a(x)T^a)$ are $N \times N$ matrices. From equation (1.5) and the cyclic property of the trace it is easy to see that $\mathrm{Tr}A_\mu \to \mathrm{Tr}A_\mu$. Accordingly, we can take the A_μ to be traceless, Hermitian field operators. As such, they can be expanded in terms of the generators of the Lie algebra, T^a, like $A_\mu = T^a A^a_\mu$. Infinitesimally, (1.5) now reads

$$A^a_\mu T^a \to A^a_\mu T^a + i\theta^a \left[T^a\,,\, T^b\right] A^b_\mu + \partial_\mu \theta^a T^a$$

or

$$A^a_\mu \to A^a_\mu - f^{abc}\theta^b A^c_\mu + \partial_\mu \theta^a \tag{1.6}$$

From (1.6), we see that up to a derivative term, the gauge field transforms under the adjoint representation of the group.

In quantum electrodynamics (QED), there is a $U(1)$ gauge symmetry. In this case, we have $U^R(x) = \exp(-ie\chi(x))$ and the familiar transformation law $A_\mu \to A_\mu - \partial_\mu \chi(x)$. Comparing with (1.6), we see that the structure constants f_{abc} must vanish, which reflects the fact that QED is an *abelian* gauge theory. This is to be contrasted with quantum chromodynamics (QCD), which has an $SU(3)$ gauge symmetry. Here a representation of the generators is given by the eight lambda matrices, and the f_{abc} are not identically zero. This is the hallmark of a *nonabelian* gauge theory.

Now we are well poised to focus on the particulars of the Standard Model. The immediate goal will be to catalogue the matter fields according to the representations they transform under in the $SU(3)$ of color, and the $SU(2) \otimes U(1)$ of electroweak theory. While a comprehensive review of the Standard Model is far beyond the scope of this survey, it will nonetheless be helpful to highlight a few salient features.

For the remainder of this work, we label the fields of the Standard Model like (R_c, D, Y) corresponding to the representations the fields belong to in $SU(3) \otimes SU(2) \otimes U(1)$, respectively. A visual synopsis of these fields is presented in figure 1. The structure of the theory

Generation	Leptons		Quarks	
1	e	ν_e	u	d
2	μ	ν_μ	s	c
3	τ	ν_τ	b	t

Bosons
γ
g
$W^\pm$ Z
h

Figure 1: The fermion (left) and boson (right) fields of the Standard Model.

allows us to consider the symmetry properties of just one generation. The other two are copies. We begin with the lepton sector. The leptons are not colored, and are thus singlets under the $SU(3)$ so $R_c = 1$. The electron, e, and its neutrino ν_e are accounted for by introducing the left handed Weyl fields ψ^l and $\bar{e}$. A left handed Weyl field is a field in the (2,1) representation of the $SU(2) \otimes SU(2)$ lie algebra locally isomorphic to that of the Lorentz group in 1+3 dimensions.

Specifically, we take ψ^l to transform like (1,2,-$\frac{1}{2}$) and $\bar{e}$ like (1,1,1). Note that because Y values add, there is no singlet (1,1,0) that can be formed by taking direct products of the representations of ψ^l and $\bar{e}$. Accordingly, the gauge invariance forbids explicit mass terms for the leptons in the Lagrangian (a Yukawa coupling with the Higgs (1,2,-$\frac{1}{2}$) is permitted, this will give mass to the fermions). Phenomenologically, we can make the identification

$$\psi^l = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \qquad \text{and} \qquad \bar{e} = \mathcal{C} e_R \tag{1.7}$$

where the L and R denote left or right handed Weyl fields, and $\mathcal{C}$ is the charge conjugation operator. Note that we have assumed the absence of a right handed neutrino.

The quark sector of the Standard Model has a somewhat similar structure. We construct an $SU(2)$ doublet of left handed fields, ψ^q, in the representation (3,2,$\frac{1}{6}$) and the left handed singlets, $\bar{u}$ ($\bar{3}$,1,$-\frac{2}{3}$) and $\bar{d}$ ($\bar{3}$,1,$\frac{1}{3}$). Here we identify

$$\psi^q = \begin{pmatrix} u_L \\ d_L \end{pmatrix} \qquad \text{and} \qquad \bar{u} = \mathcal{C} u_R \;\;,\;\; \bar{d} = \mathcal{C} d_R \tag{1.8}$$

Somewhat tangentially, it may be useful to recognize that this generation can be organized in terms of Dirac fields like

$$E = \begin{pmatrix} e \\ \bar{e}^\dagger \end{pmatrix} \qquad N = \begin{pmatrix} \nu \\ 0 \end{pmatrix} \qquad D = \begin{pmatrix} d \\ \bar{d}^\dagger \end{pmatrix} \qquad U = \begin{pmatrix} u \\ \bar{u}^\dagger \end{pmatrix} \tag{1.9}$$

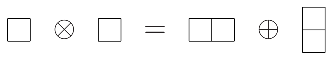
All things said and done, we see that the standard model is a gauge theory that describes three generations of left handed Weyl fields in the representation $(1, 2, -\frac{1}{2}) \oplus (1, 1, 1) \oplus (3, 2, \frac{1}{6}) \oplus (\bar{3}, 1, -\frac{2}{3}) \oplus (\bar{3}, 1, \frac{1}{3})$. By introducing a complex scalar field in the representation $(1, 2, -\frac{1}{2})$, we can construct a mechanism through which some of these fields acquire a mass (more on this in section 4). In section 3 we will search for a Lie group that contains $SU(3) \otimes SU(2) \otimes U(1)$ as a subgroup, and try to place the fields of the Standard Model into *its* representations. Section 5 explores the novel physics that this simple unification scheme requires.

2. A Few Facts About SU(5)

Any simple Lie group containing $SU(3) \otimes SU(2) \otimes U(1)$ as a subgroup must satisfy some obvious constraints. Namely, we need $2 + 1 + 1 = 4$ or more generators in the Cartan subalgebra. This narrows our search to groups of (at least) rank 4. Also, note that the representation containing the fields is complex. As it turns out, among the rank 4 groups, only $SU(5)$ has complex representations. For our first (and only) try, we shall see if we can fit these fields into representations of $SU(5)$.

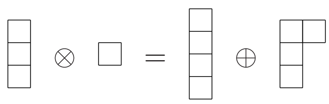
In order to make this procedure relatively painless, it will be wise to compile some pertinent information about $SU(5)$. We do this now. Specifically, we would like to understand what representations exist, whether they appear symmetrically or anti-symmetrically, and (most importantly) how these representations decompose into irreducible representations of $SU(3) \otimes SU(2) \otimes U(1)$. The first two of these are easy. We can explore the representations and their symmetry properties via Young tableaux. A five component tensor transforming under the fundamental representation is depicted by a $\square$.

The direct product of two fundamental representations is easily seen to be



(2.1)

computing the "factors over hooks" to find the dimensionality of the diagrams on the right hand side of (2.1), and noting the first is symmetric and the second is antisymmetric, one obtains $5 \otimes 5 = 15_{\mathrm{S}} \oplus 10_{\mathrm{A}}$. Also, we can do



(2.2)

which is just $\overline{10} \otimes 5 = \bar{5}_{\mathrm{A}} \oplus \overline{45}$. In passing, we note that the dimension of the adjoint representation is given by $5^2 - 1 = 24$.

We next begin the task of determining how these representations decompose into representations of the Standard Model group. A pragmatic way of doing this is to work it out for the fundamental representation, and use the products in (2.1) and (2.2) to construct further decompositions. For the 5 of $SU(5)$, this is particularly simple. Out of the 24 generators represented by traceless, Hermitian 5×5 matrices, we take

$$\begin{pmatrix} \frac{1}{2}\lambda_\alpha & 0 \\ 0 & 0 \end{pmatrix} \; (\alpha = 1, 2, \ldots, 8) \qquad \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2}\sigma_\beta \end{pmatrix} \; (\beta = 1, 2, 3) \qquad Y = \begin{pmatrix} -\frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \tag{2.3}$$

to generate the Standard Model subgroup. In (2.3) the λ_α are the Gell-Mann matrices, the σ_β are the Pauli matrices, and Y is the hypercharge. Implicitly, I have chosen the charge operator to be $Q = T_3 + Y$.

For an $SU(5)$ tensor transforming under the fundamental representation, v_i, we can arrange the indices such that the first three, v_α, are acted on by the λ_α and the final two, v_β, transform under the σ_β. Importantly, when the generators of the subgroup act on v_i, the v_α transform as singlets with respect to $SU(2)$ and the v_β transform as singlets with respect to the $SU(3)$. Since the $\bar{2} = 2$ in $SU(2)$, it is obvious that the v_β transform like $(1,2,\frac{1}{2})$ in the Standard Model subgroup. For the hypercharge chosen above, we take the v_α to transform in the 3 (not $\bar{3}$) of $SU(3)$, so in the subgroup, these transform as $(3,1,-\frac{1}{3})$. To summarize, we can write the decomposition like

$$5 \to (1,2,\frac{1}{2}) \oplus (3,1,-\frac{1}{3}) \tag{2.4}$$

To see how the 15_S and the 10_A decompose, one simply forms the direct product in (2.1) using the explicit decomposition in (2.4):

$$\begin{aligned} 15 = 5 \otimes_S 5 &\to (1 \otimes_S 1, 2 \otimes_S 2, \frac{1}{2} + \frac{1}{2}) \oplus (3 \otimes_S 3, 1 \otimes_S 1, -\frac{2}{3}) \oplus (3,2,\frac{1}{6}) \\ &= (1,3,1) \oplus (6,1,-\frac{2}{3}) \oplus (3,2,\frac{1}{6}) \end{aligned} \tag{2.5}$$

if the notation isn't obvious, the $\otimes_s$ means the symmetric part of the direct product. In much the same way, it is not hard to see that

$$10 = 5 \otimes_A 5 \to (\bar{3},1,-\frac{2}{3}) \oplus (3,2,\frac{1}{6}) \oplus (1,1,1) \tag{2.6}$$

The only other decomposition of interest is the 45. The procedure here will be to take the conjugate of representation (2.6) and form the direct product with (2.4):

$$\begin{aligned} \overline{10} \otimes 5 \to (1,2,-\frac{1}{2}) \oplus \ (\bar{3},1,\frac{1}{3}) \oplus (3,2,\frac{7}{6}) \oplus (\bar{3},3,\frac{1}{3}) \oplus (\bar{3},1,\frac{1}{3}) \oplus \ (6,1,\frac{1}{3}) \\ \oplus (8,2,-\frac{1}{2}) \oplus (3,1,-\frac{4}{3}) \oplus (1,2,-\frac{1}{2}) \end{aligned} \tag{2.7}$$

Discarding the antisymmetric decomposition of $\bar{5}$ leaves behind the representation we are after. To wit,

$$\overline{45} \to (3,2,\frac{7}{6}) \oplus (\bar{3},3,\frac{1}{3}) \oplus (\bar{3},1,\frac{1}{3}) \oplus \ (6,1,\frac{1}{3}) \oplus \ (8,2,-\frac{1}{2}) \oplus (3,1,-\frac{4}{3}) \oplus (1,2,-\frac{1}{2}) \tag{2.8}$$

finally, it will be convenient to look at how the 24 transforms. Since $5 \otimes \bar{5} = 1 \oplus 24$, we use (2.4) again to see that

$$24 \to (1,1,0) \oplus (1,3,0) \oplus (3,2,-\frac{5}{6}) \oplus (\bar{3},2,\frac{5}{6}) \oplus (8,1,0) \tag{2.9}$$

At this juncture, the mathematics necessary for this review is entirely in hand. Not only have we enumerated some useful representations of $SU(5)$, but we also know how these representations decompose into representations of $SU(3) \otimes SU(2) \otimes U(1)$. With this, and the representations of the matter fields worked out previously, it is painfully obvious how to proceed...

3. A Little Like House Keeping

From our work in sections 1 and 2, it is apparent that $SU(5)$ was a remarkable choice for an attempt at unification. Specifically, the five fields in the lepton doublet and the $\bar{d}$ singlet are exactly the representations that appear in the $\bar{5}$ of $SU(5)$. This leaves six fields in the quark doublet, three fields in the $\bar{u}$ singlet, the left handed positron, and the Higgs. The first ten of these (everything but the Higgs) is exactly the content of the antisymmetric 10, as per (2.6). For the Higgs in the representation $(1{,}2{,}-\frac{1}{2})$, lines (2.4) and (2.8) imply that we can fit it into either the $\bar{5}$ or the $\overline{45}$ representation of $SU(5)$.

Almost miraculously, we have successfully placed all the matter fields of the Standard Model into representations of $SU(5)$. Explicitly, we have the fields

$$\psi = \begin{pmatrix} \bar{d}_r & \bar{d}_b & \bar{d}_g & e & -\nu \end{pmatrix} \quad \chi = \begin{pmatrix} 0 & \bar{u}_g & -\bar{u}_b & u_r & d_r \\ -\bar{u}_g & 0 & \bar{u}_r & u_b & d_b \\ \bar{u}_b & -\bar{u}_r & 0 & u_g & d_g \\ -u_r & -u_b & -u_g & 0 & \bar{e} \\ -d_r & -d_b & -d_g & -\bar{e} & 0 \end{pmatrix} \quad H = \begin{pmatrix} \varphi_r & \varphi_b & \varphi_g & h^- & -h^0 \end{pmatrix} \tag{3.1}$$

transforming under the $\bar{5}$, 10, and $\bar{5}$ representations, respectively.

As (3.1) shows in no uncertain terms, some of these irreducible representations contain both quarks and leptons. This implies that it may be possible for our unified theory to contain interactions that violate baryon number, for example changing down quarks into positrons. In section 5 we will see that these interactions do indeed appear in the theory, leading to surprising predictions that await experimental verification.

Eventually, it will be interesting to look at the interactions the components of these fields have with the gauge bosons of the unified theory. Generalizing our discussion from section 1, this will involve replacing any derivatives in the Lagrangian with the covariant derivative, D_μ, which depends on the gauge fields, A_μ. For the left handed Weyl fields, the kinetic term is given by

$$\mathcal{L} = i\psi_i^\dagger \bar{\sigma}^\mu D_\mu \psi^i + \frac{i}{2}\chi^{\dagger ij}\bar{\sigma}^\mu D_\mu \chi_{ij} \tag{3.2}$$

where $\bar{\sigma}^\mu = (I, -\vec{\sigma})$ and the $\vec{\sigma}$ are the Pauli matrices. Here the i, j labels are indices for $SU(5)$. The covariant derivatives are easily seen to be

$$\begin{aligned} D_\mu \psi^i &= \partial_\mu \psi^i - ig_5 A_\mu^a (T_{\bar{5}}^a)^i{}_j \psi^j \\ D_\mu \chi_{ij} &= \partial_\mu \chi_{ij} - ig_5 A_\mu^a (T_{10}^a)_{ij}{}^{kl} \chi_{kl} \end{aligned} \tag{3.3}$$

of course the covariant derivative for the H is the same as for the ψ, but the kinetic term certainly is not. Furthermore, the interactions between the gauge and matter fields are mediated by one “unified” coupling, g_5. To see this, simply insert (3.3) into (3.2) and discard the ordinary derivative terms.

Now that the Standard Model has been subsumed by our $SU(5)$ GUT, we can systematically turn the crank and try to extract as much physics as possible. In fact, given a few more lines, we will have covered as much ground as Georgi and Glashow did in their original (and very enthusiastic) work [4].

The missing lines are a discussion of the relationship between our new coupling, g_5, and the Standard Model couplings g_1, g_2 and g_3. An experimentally interesting parameter is the weak mixing angle, θ_W (in electroweak theory, θ_W can be related to the mass of the Z). In terms of the coupling constants, $\tan\theta_W = g_1/g_2$. Because all interactions in the unified theory will have couplings proportional to g_5, it is possible to obtain a real prediction for the ratio of the g_1 and g_2. To do this, it is sufficient to note that all the relevant interaction terms will have a g_5, a Standard Model gauge field (normalized by its kinetic term), and a corresponding generator (either T_3 or Y). Evidently, then, the only thing that differentiates one coupling from the other is the normalization of the generator. Call N_3 and N_Y the normalization factors of T_3 and Y respectively. Since both these generators are diagonal, it is easy to see that over the fundamental representation $T_3^\dagger T_3 = |N_3|^2 \operatorname{tr} T_3^2 = 1$ so long as $N_3 = \sqrt{2}$. Similarly, we have $N_Y = \sqrt{6/5}$. In other words, for our $SU(5)$ GUT

$$\tan\theta_W = \frac{g_1}{g_2} = \frac{N_y}{N_3} = \sqrt{\frac{3}{5}} \tag{3.4}$$

Experimentally, via measurement of the Z mass (and with $\overline{MS}$ renormalization) this angle is $\theta_W = 0.502$ [5] which differs by about %24 from that calculated in (3.4).

4. Breaking it Down

There are two (related) issues that need to be resolved before we proceed. The first should be rather obvious. At the length scales currently probed by high energy physics experiments, the Standard Model is in excellent agreement with all measured results. That is to say $SU(3) \otimes SU(2) \otimes U(1)$ describes all known interactions between all known gauge bosons and all known matter fields. The fact that $SU(5)$ contains extra fields like the ϕ_a of (3.1) and predicts interactions as yet undiscovered strongly suggests that the $SU(5)$ symmetry is "spontaneously" broken down to the symmetry group of the Standard Model at some energy scale, M_{GUT}.

Additionally, we will need to develop a scheme by which the fermions acquire a mass. To recognize why this is necessary, it is sufficient to note that any term in the Lagrangian resembling a mass term for a left handed Weyl field would have to transform like $\bar{5} \otimes \bar{5} = \overline{15} \oplus \overline{10}$, or $10 \otimes 10 = \bar{5} \oplus \overline{45} \oplus \overline{50}$, or $\bar{5} \otimes 10 = 5 \oplus 45$. Since none of these direct products contain the singlet, a mass term breaks the $SU(5)$ gauge invariance. By including Yukawa terms in the Lagrangian, we will see that it is possible to break yet another symmetry and allow the fermions to become massive.

4.1 From One There Were Three

First lets break $SU(5)$ into $SU(3) \otimes SU(2) \otimes U(1)$. For reasons that will soon become clear, consider a real scalar field Φ in the adjoint representation of $SU(5)$. As mentioned in section 1, this implies a transformation like $\Phi \to \Phi + i\theta^a [T^a, \Phi]$ where $\Phi = \Phi^a T^a$ on the right hand side. Of course this also means $D_\mu\Phi = \partial_\mu\Phi - ig_5 A^a_\mu [T^a, \Phi]$. As should be familiar, if we add to our Lagrangian a potential for Φ that is minimized when $\langle\Phi\rangle = \mathfrak{v}$, in unitary gauge we can define a shifted field $\Phi(x) = (\rho(x) + \mathfrak{v})/\sqrt{2}$ and use it in any perturbative scheme we develop. It is important to realize that $\Phi, \mathfrak{v}$ and ρ are all 5×5 matrices with respect to $SU(5)$. By clever choice of symmetry transformation, we can bring $\mathfrak{v}$ into diagonal form. It remains traceless. Now something interesting happens when we use the unitary gauge expansion of Φ in the gauge invariant kinetic term $\mathrm{tr}\,(D_\mu\Phi)^2$. We have $D_\mu\Phi = D_\mu\rho - i\frac{g_5}{\sqrt{2}}A^a_\mu[T^a, \mathfrak{v}]$. Working out the kinetic term, it is easy to see that there will be a piece that looks like

$$\mathcal{L}_M = g_5^2\,\mathrm{tr}\left([T^a, \mathfrak{v}][T^b, \mathfrak{v}]\right)\; A^a_\mu A^{b\mu} \tag{4.1}$$

Note that since $A^a_\mu A^{b\mu}$ is symmetric on the indices a, b that label the gauge field, we can take 1/2 the anticommutator inside the trace if it is convenient. Obviously, when the trace doesn't vanish, the gauge fields can acquire a mass. These masses (squared) will be the eigenvalues of

$$(\mathcal{M}^2)^{ab} = -g_5^2\,\mathrm{tr}\left([T^a, \mathfrak{v}][T^b, \mathfrak{v}]\right) \tag{4.2}$$

For the case at hand, we want the unbroken gauge symmetry group to be that of the Standard Model, $SU(3) \otimes SU(2) \otimes U(1)$. It is not hard to see that this means $[T^a, \mathfrak{v}]$ must vanish for the 8 + 3 +1 = 12 generators that correspond to the Standard Model. Regarding (2.3), it is easy to see how we can ensure that this is true. Since $\mathfrak{v}$ is a diagonal, traceless, $3 + 2 \times 3 + 2$ matrix, we can take $\mathfrak{v}$ to have a 3×3 block with a along the diagonal and a 2×2 block underneath it with b along the diagonal. This requires the constraint $3a + 2b = 0$. We already know one matrix that has this form, the $SU(5)$ hypercharge generator Y. Accordingly, we can choose $\langle\Phi\rangle = \mathfrak{v} = vY$, which ensures that $\mathfrak{v}$ commutes with all the $SU(5)$ generators. Here v is a number with mass dimension one. Generically, it sets the energy scale at which the gauge invariance is spontaneously broken.

As suggested above, the broken generators (those that do not commute with $\mathfrak{v}$ are capable of giving mass to the remaining twelve gauge bosons. Using the decomposition in (2.9), we see that three of the five terms correspond to the gauge bosons of the standard model: (8,1,0) for the eight gluons, (1,3,0) for the three "W" (remember we have not yet broken $SU(2){\otimes}U(1)$) and (1,1,0) for the B. This implies that the twelve broken generators, X, transform under $(3, 2, -\frac{5}{6}) \oplus (\bar{3}, 2, \frac{5}{6})$. These representations are clearly conjugates of one another, so we can take the X to transform as a *complex* field under (3,2,-$\frac{5}{6}$). Now we can use (4.2) to find the mass of the twelve X. To minimize work, we should find a generator of $SU(5)$ that is not one of those in (2.3). One of them, taken from [6], is

$$T^4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.3}$$

since $[T^4, \mathfrak{v}][T^4, \mathfrak{v}] = -v^2\frac{25}{144}[\delta_i{}^1\delta^j{}_1 + \delta_i{}^4\delta^j{}_4]$ we find at once that the mass of the X is given by $M_X = g_5 v\frac{5}{6\sqrt{2}}$.

The massive X are one of the most interesting features of $SU(5)$ grand unification. As per (2.3), the five values an $SU(5)$ index can take are divided into $i \to c, w$ where

$c = 1, 2, 3$ and $w = 4, 5$. In this notation, the eight gluons are the components $A_c{}^{c'}$ (with the constraint $A_c{}^c = 0$), and the three W are the $A_w{}^{w'}$ (with the constraint $A_w{}^w = 0$). The twelve X, however, have split indices: $X_w{}^c$ and $X_c{}^w$. Thus they mediate interactions between the Standard Model fields that are the c components of the $\bar{5}$ and the fields that are the w components, for example. Although we will make this (a bit) more precise in section 5, we can already see that the absence of these interactions in today's high energy experiments suggests that the mass of these X must be larger than the few TeV scale.

4.2 How to Make a Fermion Fat

As we discovered earlier, there is still the problem that none of our fermion fields (which are in the $\bar{5}$ and the 10 of $SU(5)$) are allowed (by gauge invariance) to have mass. This remains true in the unbroken $SU(3) \otimes SU(2) \otimes U(1)$. That said, it has been known since [7] that there is a trick for circumnavigating this apparent show stopper.

It will be nice to spend a few lines reviewing how this works in the Standard Model. Experimentally, the $W^{\pm}$ and the Z are massive, while the photon is not. This implies that in a way very much analogous to what was done above, the $SU(2) \otimes U(1)$ of electroweak theory is spontaneously broken. In the process, the gauge bosons of the weak interaction acquire mass, but the photon does not.

In electroweak theory, one can make the empirical observation that $(2, -\frac{1}{2}) \otimes (2, -\frac{1}{2}) \otimes (1, 1) = (1, 0) \oplus (3, 0)$. The singlet, (1,0) appears, so if we take our Higgs field φ to transform as $(2, -\frac{1}{2})$ we can make a gauge invariant interaction in the Lagrangian like

$$\mathcal{L}_Y = -y\phi_1 e\bar{e} + y\phi_2 \nu\bar{e} + \text{h.c} \tag{4.4}$$

in equation (4.4) y is a coupling for the interaction, and I have used the fact that the antisymmetric 1 in $SU(2)$ implies that ϵ^{ij} is an invariant symbol we can use to contract $SU(2)$ indices.

Just as before, if the potential for φ is minimized by $\langle\varphi\rangle = \mathfrak{v}$, in the unitary gauge $\varphi = (\rho + \mathfrak{v})/\sqrt{2}$. By rotating $\mathfrak{v}$ with a symmetry transformation, we can take $\mathfrak{v}_1 = v$ and $\mathfrak{v}_2 = 0$. This in turn lets us rewrite the interaction term (4.4) with the Dirac fields in (1.9) like

$$\mathcal{L}_Y = -\frac{y}{\sqrt{2}}(v + \rho)\bar{E}E \tag{4.5}$$

From the above, it is easy to see that the electron has acquired a mass term, and $m_e = \frac{yv}{\sqrt{2}}$. Without too much imagination, it is possible to see how this generalizes for the quark sector.

This recap should be sufficient to apply the mechanism to our unified theory. We have already seen from (2.8) and (2.4) that the $\overline{45}$ and $\bar{5}$ both contain the desired representation (1,2,-$\frac{1}{2}$). Here I will choose the Higgs in (3.1), that is, the one that transforms under the $\bar{5}$. Notice that I have used the charge operator $Q = T_3 + Y$, to label the last two components of H by their electric charge.

Earlier in this section, we found that $10 \otimes 10 = \bar{5} \oplus \ldots$ and $\bar{5} \otimes 10 = 5 \oplus \ldots$ Since $\bar{5} \oplus 5 = 1 \oplus 24$, it is obvious that we can make gauge invariant Yukawa type interaction

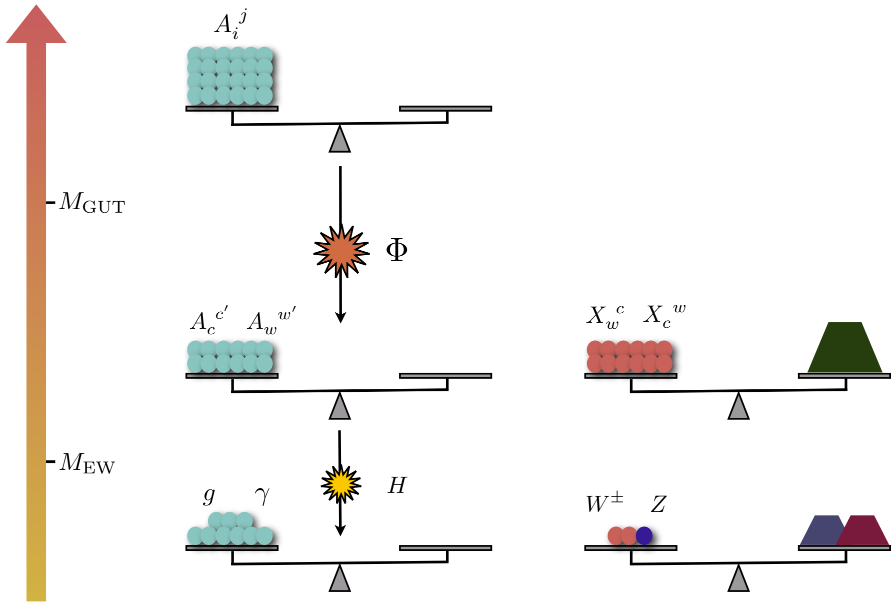


Figure 2: Cartoon of spontaneous symmetry breaking in an $SU(5)$ grand unified theory. The leftmost arrow shows the energy scale at which the symmetry is broken. The spheres represent gauge bosons "weighed" on scales. Only the teal subsets of the original 24 massless gauge bosons remain massless. Naming conventions match those in the text. Experimentally, $M_{\rm EW} \sim 240$ GeV while $M_{\rm GUT} \sim 10^{12}$ GeV.

terms with H and $H^\dagger$ just as we did with ϕ in the electroweak theory:

$$\mathcal{L}_Y = -yH^i\psi^j\chi_{ij} - \frac{z}{8}\epsilon^{ijklm}H_i^\dagger\chi_{jk}\chi_{lm} \tag{4.6}$$

Once again, the fact that the singlet appears antisymmetrically in $SU(5)$ means the ϵ tensor totally antisymmetric on five indices is an invariant symbol for terms in our GUT's Lagrangian.

If one so desired, it would not be difficult to write (4.6) in terms of the fields of the Standard Model. Since the result is somewhat messy, it won't be worth much to write it explicitly. It is, however, worthwhile to observe that all Standard Model fields couple to both the colored components of H, ϕ_c as well as the φ^- and φ^0. Those coupled to the latter can lead to mass terms for the fermions after another bout of spontaneous symmetry breaking. Those coupled to the former will form interaction terms that violate baryon number conservation.

The highlights of this section are summarized in figure 2. As the cartoon suggests, it is a bit more involved to break $SU(5)$ down to the physics at our energy scale, as compared to breaking the electroweak theory. Namely, the procedure involves two different Higgs fields, transforming under different representations of $SU(5)$. Furthermore, the litany of as yet undetected massive gauge bosons and scalar fields require some explaining, which we turn to next.

5. Dealing With the Consequences

Structurally, (that is to say group theoretically) our $SU(5)$ unified theory looks pretty good. The Standard Model fields are all there, and there is a mechanism in place for breaking the gauge symmetry down to the $SU(3) \otimes U(1)$ gauge symmetry present at "ordinary" energies. That said, it remains to discover whether or not the unified theory is a useful description of fundamental physics. In this vein, there are two obvious questions we need to answer:

1. Does this theory tell us anything about our world we can't already explain?
2. If so, does this theory make any novel predictions that can be verified experimentally?

As one might imagine, there are many ways to answer these questions, with varying degrees of sophistication. In this section we will provide at least a cursory answer to both.

5.1 Neutral Hydrogen and Quantized Charge

Some of the most fundamental observations a physicist can make also prove to be the most difficult to describe theoretically. This is certainly the case for the experimental fact that the charge of the electron is exactly opposite the charge of the proton, and that this charge is quantized. In the context of our $SU(5)$ unified theory, both of these peculiarities emerge naturally, and in an almost trivial way.

Consider first the quantization of the electric charge. As we have seen, the gauge fields of $SU(5)$ couple to the 24 generators of the group. This is obvious from the form of the covariant derivative, where $D_\mu = \partial_\mu - ig_5 T^a A^a_\mu$ implies that the combination $g_5 T^a$ behaves as a sort of charge that the gauge boson A^a couples to. Since the photon in $SU(3) \times U(1)$ couples to electromagnetic charge $Q = T^3 + Y$, there is some linear combination of the A^a that represents the photon. This combination must correspond to generators that remain massless after the spontaneous symmetry breaking(s).

In the unified theory, the defining representation can be described by generators written as 5×5 matrices. This of course includes those that correspond to T^3 and Y. This is a consequence of that fact that $SU(5)$ is a simple Lie group, and thus Y is not the generator of a $U(1)$ product. This should be contrasted with the case of $SU(3) \otimes SU(2) \otimes U(1)$ where Y is some number. As a generator of $SU(5)$, Y is constrained to be traceless and Hermitian, so it can no longer have arbitrary components. From this, it is easy to see that Q in *any* simple group will be Hermitian with zero trace as well. Diagonalizing Q, we find that the charge takes on quantized values, and that the sum of the charges of particles in a given representation must vanish.

As a specific example, it is easy to see that for the $\bar{5}$ of (3.1), Q is given by

$$Q = T^3 + Y = \begin{pmatrix} -\frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.1}$$

and thus $3Q_{\bar{d}}+Q_e = 0$, or $Q_{\bar{d}}/Q_e = -1/3$. If Q_e = -1, then $Q_{\bar{d}} = 1/3$. Amazingly, the ratio of these charges has emerged as a consequence of the theory, rather than an experimentally motivated input. Now we can compute the charge of the proton and find out whether or not hydrogen is electrically neutral in an $SU(5)$ universe. Since left and right handed fields are related through charge conjugation, $Q_d = -Q_{\bar{d}}$. Additionally, we can show that $Q_u = T_u^3 + Y_u = 1/2 + 1/6 = 2/3$, and thus $Q_p = 4/3 - 1/3 = 1$ just as hoped. The fact that the charge of the proton is opposite, but equal in magnitude to that of the electron has been verified to outstanding precision [5].

Although it is beyond the scope of this review, the fact that charge is quantized in our $SU(5)$ unified theory can also be viewed as a consequence of the existence of a t'Hooft-Polyakov monopole in the theory. This magnetic monopole appears in spontaneously broken gauge theories, and through a familiar (but somewhat involved) argument, its existence implies the quantization of electric charge. Ignoring the details, it turns out that the mass of this magnetic monopole is somewhat larger than the M_X of section 4. This convenient observation helps justify the fact that these monopoles have not yet been observed in high energy particle physics experiments.

5.2 The Death of the Proton

Without doubt, one of the most salient features of our GUT is the appearance of massive gauge bosons that mediate interactions which turn quarks into leptons (and vice-versa). This peculiarity first arose as a consequence of breaking $SU(5) \to SU(3) \otimes SU(2) \otimes U(1)$, and was discussed briefly in section 4. The possibility of decays that disrespect baryon number conservation is very exciting from a cosmological standpoint. As is well known, our universe is full of ordinary matter. Since we assume that matter and antimatter were created in equal amounts, the fact that we don't observe "anti-asteroids", or better yet, the fact that we observe *anything* at all presents a bit of a puzzle. For the initial symmetry between matter and antimatter to be broken, it must be that the relevant physics differentiates between particles and anti-particles (like the weak force) and baryon number can change through an allowed interaction. With these conditions satisfied, one can develop a simple scheme by which a universe with zero initial baryon number acquires a very small (but non-zero!) baryon number: If the length scale characteristic of the early universe is below $1/M_{\text{GUT}}$, the gauge bosons that mediate baryon number violating interactions are copiously produced. The exchange of these bosons is then likely to alter the net baryon density in the universe, and may be capable of creating the observed baryon to photon ratio $N_B/N_\gamma \sim 10^{-10}$.

In keeping with the tone of this review, we will now explore the phenomenon of proton decay from a perspective that favors group theory arguments over explicit computation. The aim will be to use what we have learned to estimate the lifetime of the proton. Although (relatively) straightforward, explicit calculation of this decay rate is a fairly technical exercise. Here it will suffice to sketch the procedure, and highlight the results.

Before doing any real work, lets recap a few important facts. When the $SU(5)$ gauge symmetry is broken by the scalar Φ, 12 of the 24 generators break, and the corresponding

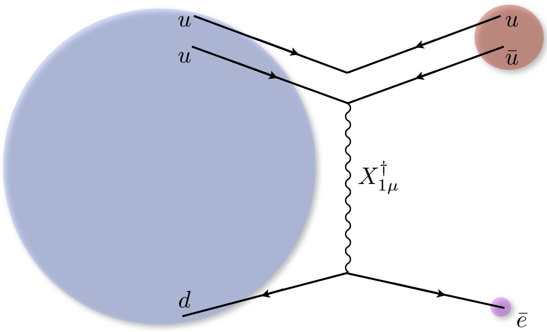


Figure 3: Proton decay as mediated by the X. Initially, $B - L = 1$, and this remains true after the decay. Neither B nor L are independently conserved.

gauge bosons acquire a mass $M_X = g_5 v \frac{5}{6\sqrt{2}}$. Generically, these gauge bosons carry "split" indices with respect to the gauge group, ${X_w}^c$ or ${X_c}^w$ where $c = 1, 2, 3$ and $w = 4, 5$.

From the gauge invariant kinetic term for the fields in the $\bar{5}$ and 10 (3.2), it is easy to see that the interaction term describing the exchange of an X between fields in the 10 is

$$\mathcal{L}_{int} = -g_5 \,\mathrm{tr}\left(\chi^\dagger X_\mu \bar{\sigma}^\mu \chi\right) \tag{5.2}$$

An experimentally accessible channel for proton decay is $p \to \bar{e}\pi^0$. In terms of fundamental particles, this decay looks like $uud \to u\bar{u} + \bar{e}$. From this, we see that the relevant term in (5.2) must be

$$\mathcal{L}_{p\to\bar{e}\pi^0} = \frac{g_5}{\sqrt{2}}\left(X^{\dagger c}_{1\mu}\bar{e}^\dagger\bar{\sigma}^\mu d_c - \epsilon_{abc}X^{\dagger a}_{1\mu}u^{\dagger b}\bar{\sigma}^\mu \bar{u}^c\right) + \text{h.c.} \tag{5.3}$$

which allows for $u \to \bar{u}$ and $d \to \bar{e}$. This decay is illustrated diagrammatically in figure 3. Notice that while baryon number and lepton number are independently violated, the quantity $B - L$ is conserved.

From the figure and a few elementary considerations from field theory, it is already possible to obtain an estimate for the decay rate of the proton. Each vertex introduces a factor proportional to g_5, and (at least for sufficiently large M_X) we can take the internal vector boson propagator to go like $1/M_X^2$. Very roughly, then, the diagram of figure 3 gives an invariant amplitude $\mathcal{M} \sim (g_5/M_X)^2$. In the center of mass frame, the decay rate is generically given by the product of $|\mathcal{M}^2|$ and a phase space factor. The scale of the decay is set by the mass of the proton, m_p, so we can just multiply the square of our invariant amplitude by however many m_p's we need to get units of inverse time. This approach gives $\Gamma \sim m_p^5 (g_5/M_X)^4$. In terms of the GUT symmetry breaking scale v, the lifetime of the proton is then easily seen to be

$$\tau = \frac{1}{\Gamma} \sim \frac{1}{4m_p^5}\left(\frac{5\,v}{6}\right)^4 \tag{5.4}$$

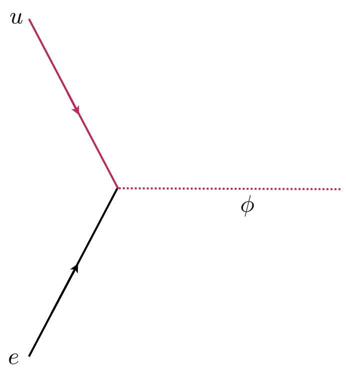


Figure 4: An allowed interaction between standard model fields and the heavy component of H. These types of interactions can be combined to predict processes which violate baryon number conservation.

Interestingly (but not so surprisingly), the larger the energy scale at which the unified theory breaks, the longer the proton lives. Ongoing experiments suggest that if the proton decays at all, it takes an astoundingly long time for it to do so—maybe on the order 10^{33} years according to [5]. This would imply $v \sim 10^{10}$ GeV. A more careful analysis, based on renormalization group flows, for example, pushes this about 5 orders of magnitude higher.

The exchange of the X is not the only means by which a proton might decay. Indeed, in section 4 we discovered that the Higgs field responsible for breaking the spontaneous symmetry of the Standard Model gauge group, H, transforms under the $\bar{5} \to (3, 1, -\frac{1}{3}) \oplus (1, 2, \frac{1}{2})$. Clearly, there is a triplet component with respect to $SU(3)$—these are the ϕ_c. With a few lines of algebra, it is easy to see that the Yukawa term in (4.6) involving these fields is given by

$$\mathcal{L}_{Y_\phi} = -y\epsilon^{ij}\phi^a q_{ai} l_j - y\epsilon_{abc}\phi^a \bar{d}^b \bar{u}^c - z\phi_a^\dagger \bar{u}^a \bar{e} + \text{h.c.} \tag{5.5}$$

For illustration, a typical term in this Lagrangian is depicted in figure 4. Now we have a problem. Just as before, the fact that we have never observed a decay like this implies that if our $SU(5)$ theory is a good one, there is a lower limit on the mass of the ϕ, around 10^{10} GeV. This is far larger than the scale of electroweak breaking ($\sim$ 100 GeV) which is supposed to be accomplished by the doublet component of H, φ_i. The fact that ϕ and φ sit in the same representation makes it natural to assume that $m_\phi \sim m_\varphi$. The experimentally motivated observation that these masses differ by something like eight orders of magnitude is a noteworthy theoretical challenge, and is generically referred to as the "doublet-triplet problem".

6. When All is Said and Done

By now, there exist all sorts of unified theories based on all sorts of groups, that place the

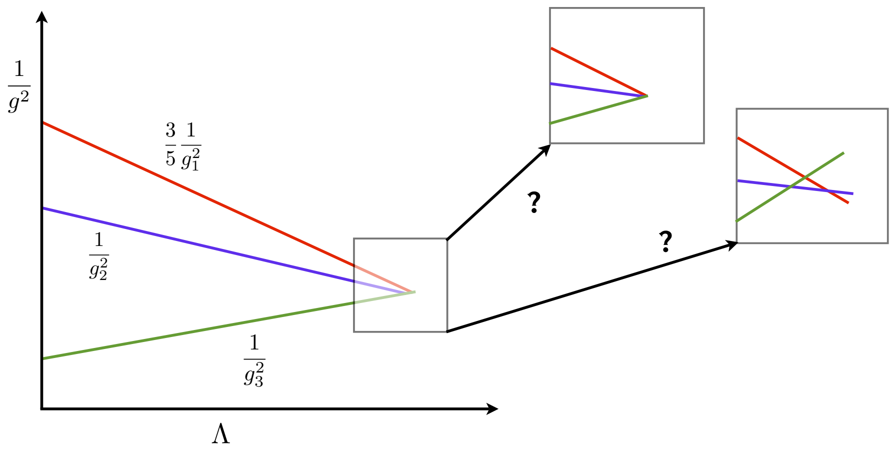


Figure 5: As the coupling constants of the Standard Model run, do they meet at one point or narrowly miss one another?

matter fields of the Standard Model into all sorts of representations. Some of them, like $SO(10)$, usurp $SU(5)$ in the sense that the all of the matter fields can be placed into a single representation of the group. Others contain exotic particles (so far unobserved) that are capable of prolonging the life of the proton.

Some of the most interesting GUTs under consideration contain bosonic counterparts for each of the fermions. These suspersymmetric theories are particularly interesting because they are capable of tweaking the renormalization group flows such that for some energy scale, M_{GUT}, $g_1 = g_2 = g_3$. Very briefly, the idea is that the particle content of a theory is capable of influencing the dependence of the the coupling, g, on the energy scale, Λ, at which it is measured. This is familiar from quantum chromodynamics, where the beta function to one loop is

$$\beta(g) \equiv \frac{d\,g}{d\log\Lambda} = -\frac{g^3}{16\pi^2}\left(11 - \frac{2}{3}N_f\right) \tag{6.1}$$

Evidently, adding more quark flavors N_f can have dramatic consequences for the running g. Specifically, as soon as the number of flavors is larger than 17, the beta function of QCD changes sign from negative to positive. Physically, this means that the theory is no longer asymptotically free in the ultraviolet.

The running of the coupling constants in the Standard Model has been studied extensively. As is well known, as the energy scale increases, the couplings seem to flow towards a common value. If there exists some scale $M_{\rm GUT}$ at which all three couplings meet at a point, it would suggest that our unification idea is not completely crazy. This is sketched in figure 5. As it turns out, in the Standard Model these couplings do *not* appear to simultaneously intersect. In a way very much like that discussed above, adding supersymmetric fields into the mix can (with some work) alter the flows and force the couplings to unify. It is for this reason that many unified theories currently considered for phenomenology include supersymmetry as an essential ingredient.

All things considered, the idea of a "grand unification" of the strong, weak, and electromagnetic forces is immensely attractive. The discovery that we *can* construct such a unification, based on a simple Lie group, is both surprising and encouraging. The fact that it is capable of producing testable predictions is better yet. As we have seen, these theories have all sorts of interesting features, like proton decay and magnetic monopoles. For better or worse, the failure of experimentalists to observe these features highly constrains the theory. Presently, experimental bounds on the proton's lifetime likely rule out simple unified theories based on $SU(5)$. It remains to be seen whether or not the other GUTs, with or without exotic modifications, suffer the same fate.

Acknowledgments

A good discussion of the use of Lie groups in physics is [8]. An accessible review of the Standard Model, and some GUT preliminaries, can be found in [6, 9]. Many of the conventions used here match the latter.

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