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Bosonic Realization of a Universal W-Algebra and Z_∞ parafermions

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Abstract

We construct a field theoretic representation of the universal W-algebra proposed by Pope, Romans and Shen, using a free complex boson in two dimensions. The resulting symmetry algebra is generated by conformal fields with spin 2,3,4,.... and has central charge $c = 2$. Highest weight representations are also given in terms of vertex operators. Furthermore, we discuss the relation of this representation to the theory of Z_∞ parafermions .

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1. **Introduction**

The notion of W-algebras was introduced first by Zamolodchikov in the context of 2-d conformal field theories that possess additional symmetries generated by a collection of higher spin, chiral fields $\{W^s(z)\}$, [1]. Despite the complicated nature of their commutation relations, W-algebras have provided a unifying conceptual framework for applying the bootstrap (operator algebra) approach to a large class of Rational Conformal Field Theories (RCFTs) that describe 2-d critical statistical models. The simplest example in the class of extended bosonic conformal symmetries is the W_N algebra generated by the stress tensor $T(z)$ and the chiral fields $\{W^s(z)\}$ with integer spin $s = 3, 4, 5, ..., N$. In this case (and for any fixed value of N) there is a series of unitary CFT models parametrized by a positive integer $p = 1, 2, 3, ...$ which are minimal in the sense that the corresponding number of representations of W_N is finite, [2]. These theories are Z_N symmetric and have central charges,

$$c_p^N = (N-1)\left[1 - \frac{N(N+1)}{(N+p)(N+p+1)}\right] = 2p + O(\frac{1}{N}) \ . \tag{1.1}$$

The theories above can be identified with the coset models,

$$\frac{SU(N)_1 \otimes SU(N)_p}{SU(N)_{p+1}} \ . \tag{1.2}$$

The construction of all unitary models associated with the large N limit of W-algebras poses an interesting problem in quantum field theory. Its solution might also be of some value in mathematics because as we will see later, W_∞ algebras are intimately related with area-preserving diffeomorphisms of 2-manifolds. According to the sequence (1.1), which has no upper bound in the limit $N \to \infty$, the only unitary representations of W_∞ (if they exist as well defined quantum field theories) will occur with central charge, $c = 2, 4, 6, ...$ However the number of conformal blocks becomes infinite in that limit and the concept of minimality is not obviously helpful for solving the models. Moreover for large N, the spectrum of anomalous dimensions collapses into three disjoint sets, some tend to zero, some others to finite values and the remaining to infinite values. It is the purpose of the present work to investigate the structure of the W_∞ algebra using a field theoretic representation which is appropriate for describing the simplest model with $c = 2$. As a byproduct we will obtain a class of highest weight (hw) representations for W_∞.

The interest in the large N behavior of W_N algebras arose from the observation that W_∞ is a deformation of the infinite dimensional symmetry algebra of area-preserving diffeomorphisms , [3]. The commutation relations of the latter are of the form,

$$[W_m^s, W_n^{s'}] = ((s'-1)m - (s-1)n)W_{m+n}^{s+s'-2} \tag{1.3}$$

where both $s, s' \geq 2$ and $m, n \in Z$. This algebra can be represented by the Poisson bracket of functions $W_n^s = x^{n+s-1} y^{s-1}$ on the two dimensional plane with (canonical) coordinates x and y or equivalently by the smooth functions $e^{inx} y^{s-1}$ on the cylinder, $R \times S^1$. At any rate, the structure (1.3) describes the leading highest spin contribution to the commutation relations of W_N at large N, provided that W_n^s are identified with the Fourier modes of the generating conformal fields $W^s(z)$ and $W^2(z) = T(z)$. It should be emphasized that the identification is purely algebraic and the area-preserving diffeomorphism symmetry does not reflect the geometry of the 2-d world of CFTs. If that were the case, (chiral) conformal transformations would be incompatible with area-preserving diffeomorphisms . For this reason it is appropriate to introduce an auxiliary surface (membrane) to interpret (1.3) geometrically.

The complete structure of W_∞ may be described as a deformation of the symmetry algebra (1.3). In particular, the results of [3] suggest that for any given s and s', the commutation relations of the area-preserving diffeomorphism algebra and W_∞ differ from each other by local functionals of the generating fields with spin less than $s + s' - 2$. Since both infinite dimensional algebras satisfy the Jacobi identity (associativity), the deformation terms cannot be arbitrary; they are 2-cocycles of the algebra (1.3) with non-trivial coefficients in general. There have already been several results in the literature concerning the nature of these terms, [3,4,5], but at the moment there does not seem to be a unique answer. This is not very surprising because as $N \to \infty$, the limit of W_N might not be uniquely defined.* Different limiting procedures can give rise to inequivalent expressions for the 2-cocycle terms, while the leading structure (1.3) remains unchanged. † Algebraically, there is no a priori way to single out one deformation from the others. It is the representation theory of the symmetry algebra in question that will provide a definite realization for the commutation relations of W_∞. This is our main motivation for being interested in a definite (perhaps the simplest) bosonic field theoretic representation of W_∞ with $c = 2$. Of course the CFT model we have in mind is the theory of Z_N parafermions in the limit $N \to \infty$.

In section 2 we discuss a realization of the W_∞ algebra proposed in [5] in terms of a free complex boson with $c = 2$. In section 3 we present a class of highest weight unitary irreducible representations of the universal W_∞ algebra. In section 4 we elucidate the relation of this realization with the limit of the Z_N parafermionic theory as $N \to \infty$. Finally, section 5 contains our conclusions and further comments.

2. A bosonic realization of the PRS W_∞ algebra

*We thank E. Witten for emphasizing this point repeatedly.

†Large N limits come in at least three distinct classes, as suggested by the Murray-von Neumann theory of factors.

The starting point in what follows, is the linear deformation of the area-preserving diffeomorphism algebra described by,

$$[W^s_m, W^{s'}_n] = ((s'-1)m-(s-1)n)W^{s+s'-2}_{m+n} + q^{2(s-2)}c_s(m)\delta_{s,s'}\delta_{m+n,0}+$$
$$+q^2 g_2^{ss'}(m,n)W^{s+s'-4}_{m+n} + q^4 g_4^{ss'}(m,n)W^{s+s'-6}_{m+n} + \cdots \tag{2.1}$$

where the sequence of .... terms terminates with W^2_{m+n} if both s, s' are either even or odd and with W^3_{m+n} otherwise. Here,

$$c_s(m) = \frac{c}{2}m(m^2-1)(m^2-4)...(m^2-(s-1)^2)\frac{2^{2(s-3)}s!(s-2)!}{(2s-1)!!(2s-3)!!} \tag{2.2}$$

with c being the value of the central charge in the Virasoro subalgebra and,

$$g^{ss'}_{2r}(m,n) = \frac{\varphi^{ss'}_{2r}}{2(2r+1)!}N^{ss'}_{2r}(m,n) \tag{2.3}$$

where,

$$\varphi^{ss'}_{2r} = \sum_{k=0}^{r}\frac{(-\frac{1}{2})_k(\frac{3}{2})_k(-r-\frac{1}{2})_k(-r)_k}{k!(-s+\frac{3}{2})_k(-s'+\frac{3}{2})_k(s+s'-2r-\frac{3}{2})_k}\ , \tag{2.4}$$

$$N^{ss'}_{2r}(m,n) = \sum_{k=0}^{2r+1}(-1)^k\binom{2r+1}{k}(2s-2r-2)_k[2s'-k-2]_{2r+1-k}\cdot$$
$$\cdot[s-1+m]_{2r+1-k}[s'-1+n]_k\ . \tag{2.5}$$

In the formulae above the symbols $(a)_k$ and $[a]_k$ denote

$$(a)_k \equiv a(a+1)(a+2)...(a+k-1)\ , \tag{2.6a}$$

$$[a]_k \equiv a(a-1)(a-2)...(a-k+1) \tag{2.6b}$$

with $(a)_0 = [a]_0 = 1$.

This algebra was first introduced in [5] from purely algebraic considerations and its compatibility with the Jacobi identities was verified (to a great extend) with the aid of symbolic manipulations. The linear structure of the commutation relations (2.1) is consistent with the expectation that the symmetry algebra of higher spin theories (in any number of space-time dimensions) will involve no quadratic or higher polynomial terms if we include the generators of all transformations with spin $s \geq 2$. In more physical terms this means that consistent gauge interactions of higher spin massless fields become possible when an infinite tower of fields with all possible (integer) values of spin is introduced . In this sense, (2.1) describes a particular large N limit of the chiral operator algebra W_N. We will refer to this limit as the universal W-algebra of Pope, Romans and Shen (PRS). The main task is to construct an explicit representation of it with central charge $c = 2$. Later we will see that our representation has a natural interpretation (and in fact it was motivated by) the theory of Z_∞ parafermions.

Let us consider a massless free complex boson $\phi(z)$ in 2-d with a two-point function normalized as follows,

$$\langle\phi(z)\phi(w)\rangle = \langle\bar{\phi}(z)\bar{\phi}(w)\rangle = 0\,,\ \langle\phi(z)\bar{\phi}(w)\rangle = -\log(z-w)\ . \tag{2.7}$$

The standard stress tensor of the theory is,

$$W^2(z) \equiv T(z) = -:\partial_z\phi\partial_z\bar{\phi}: \tag{2.8}$$

and it has the standard operator product expansion (OPE),

$$T(z)T(w) = \frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + \cdots \tag{2.9}$$

where $\cdots$ stand for non-singular terms. ¿From now on these terms will be ignored consistently because they do not contribute to the commutation relations. Also, $:\ :$ stands for minimal normal ordering (subtraction of the pole). Clearly, the central charge in (2.9) is $c = 2$ and the Fourier modes‡ W^2_n of the stress tensor satisfy the Virasoro algebra,

$$[W^2_m, W^2_n] = (m-n)W^2_{m+n} + \frac{1}{6}m(m^2-1)\delta_{m+n,0}\ . \tag{2.10}$$

Next we extend this representation to the full PRS algebra by introducing the following ansatz,

$$W^s(z) \equiv B(s)\sum_{k=1}^{s-1}(-1)^k A^s_k :\partial^k_z\phi\partial^{s-k}_z\bar{\phi}: \tag{2.11}$$

for all $s \geq 2$. The coefficients A^s_k and $B(s)$ are positive numbers that will be calculated shortly. However before we proceed any further, a few remarks are in order. It is obvious from dimensional analysis that the operators in (2.11) have scaling dimension s, although they are not necessarily primary fields. Also the operators in (2.11) involve only derivatives of the $U(1)\otimes U(1)$ currents, $\partial\phi$ and $\partial\bar{\phi}$ and no polynomial powers. Therefore, the ansatz (2.11) is bound to produce an operator algebra with linear determining relations. Moreover, due to the presence of the alternating sign $(-1)^k$ in (2.11), the operators $W^s(z)$ will be even (odd) under the interchange $\phi \leftrightarrow \bar{\phi}$ for s even (odd), provided that $A^s_{s-k} = A^s_k$. In view of this symmetry, we expect that the OPE of W^s with $W^{s'}$ will involve $W^{s''}$ with $s'' = s+s'-2, s+s'-4, s+s'-6, \cdots$ only, which is the main feature of the PRS algebra.

With these explanations in mind we proceed to calculate the coefficients A^s_k. Notice that there is no central term in eq. (2.1) if $s \neq s'$ (which means that W^s is a quasiprimary operator). Since we assume that the universal W-symmetry is unbroken, i.e. $\langle W^s(z)\rangle = 0$ for all s, we have

$$\langle W^s(z)W^{s'}(w)\rangle \sim \frac{\delta_{s,s'}}{(z-w)^{s+s'}}\ . \tag{2.12}$$

‡The Fourier modes of $W^s(z)$ are defined as $W^s(z) \equiv \sum_{n\in Z} W^s_n z^{-n-s}$.

This equation alone is sufficient to determine the form of A_k^s uniquely up to an overall normalization constant which we denoted by $B(s)$ in (2.11). Explicit calculation shows that,

$$\langle W^s(z)W^{s'}(w)\rangle = B(s)B(s')\frac{I(s,s')}{(z-w)^{s+s'}} \tag{2.13}$$

with,

$$I(s,s') = (-1)^s \sum_{k=1}^{s-1}\sum_{l=1}^{s'-1}(-1)^{k+l}(s+l-k-1)!(s'+k-l-1)!A_k^s A_l^{s'} \ . \tag{2.14}$$

The condition $I(s,s')=0$ for $s\neq s'$, puts severe constraints on the form allowed for A_k^s. We find that there is a unique solution to these conditions (up to normalization) given by,

$$A_k^s = \frac{1}{(s-1)}\binom{s-1}{k}\binom{s-1}{s-k} \ . \tag{2.15}$$

Notice that this solution enjoys the desirable property, $A_k^s = A_{s-k}^s$ and yields $A_1^2 = 1$[§].

Some explanation is required about the general proof of our statement. To establish the desired result it is sufficient to show that $I(s,s')=0$ for $s'=2,3,...,s-1$ and for all $s>2$. Using (2.15), we have that $I(s,2) \sim \sum_{k=1}^{s-1}(-1)^k k!(s-k)!A_k^s \sim \sum_{k=1}^{s-1}(-1)^k k\binom{s-1}{k}$, which is identically zero for $s>2$. This is easily shown using the binomial expansion of $(1-x)^{s-1}$, differentiating once with respect to x and setting $x=1$ at the end of the calculation. One can prove the rest of the identities $I(s,s')=0$ with $3\leq s'\leq s-1$ in a similar way. We only point out that for general s' the binomial expansion of $(1-x)^{s-1}$ must be differentiated at least $s'-1$ times before setting $x=1$. Extending this process to $s'=s$ we arrive at a (non-trivial) identity that will help us normalize appropriately W^s. In particular, the following is true for all $s\geq 2$,

$$I(s,s) = \frac{(2s-2)!}{s(s-1)} \ . \tag{2.16}$$

We reach an agreement with the standard normalization of the central terms in the commutation relations of the PRS algebra provided that,

$$B(s) = q^{s-2}\frac{2^{s-3}s!}{(2s-3)!!} \ . \tag{2.17}$$

Then the ansatz (2.11) yields an infinite tower of higher spin fields of the form,

$$W^2(z) = - :\partial\phi\partial\bar{\phi}: \ , \tag{2.18a}$$

$$W^3(z) = -2q :(\partial\phi\partial^2\bar{\phi} - \partial^2\phi\partial\bar{\phi}): \ , \tag{2.18b}$$

[§]The coefficients A_k^s appear in the definition of the (1,1) Jacobi polynomials, $P_{s-2}^{(1,1)}(x)$. The value of $I(s,s')$ in (2.14) follows directly from the orthogonality relations of these polynomials.

$$W^4(z) = -\frac{16q^2}{5} : (\partial\phi\partial^3\bar{\phi} - 3\partial^2\phi\partial^2\bar{\phi} + \partial^3\phi\partial\bar{\phi}) : , \tag{2.18c}$$

$$W^5(z) = -\frac{32q^3}{7} : (\partial\phi\partial^4\bar{\phi} - 6\partial^2\phi\partial^3\bar{\phi} + 6\partial^3\phi\partial^2\bar{\phi} - \partial^4\phi\partial\bar{\phi}) : , \tag{2.18d}$$

$$W^6(z) = -\frac{128q^4}{21} : (\partial\phi\partial^5\bar{\phi} - 10\partial^2\phi\partial^4\bar{\phi} + 20\partial^3\phi\partial^3\bar{\phi} - 10\partial^4\phi\partial^2\bar{\phi} + \partial^5\phi\partial\bar{\phi}) : \tag{2.18e}$$

etc. Here, q is the deformation parameter of PRS. It is worth pointing out that only W^3 is a primary conformal field.

Next, we will show that the expressions we have for $W^s(z)$ with A^s, $B(s)$ given by (2.15) and (2.17) respectively, not only reproduce the correct form of the central terms in (2.1) but also provide a representation of the full PRS algebra with $c = 2$. Before we delve into the general case, let us present the result of a sample (yet non-trivial) OPE calculation: using (2.18) and (2.7) we find that,

$$W^3(z)W^4(w) = q^2\frac{64\cdot 12}{5}\left[\frac{W^3(w)}{(z-w)^4} + \frac{1}{3}\frac{\partial W^3(w)}{(z-w)^3} + \frac{1}{14}\frac{\partial^2 W^3(w)}{(z-w)^2} + \frac{1}{84}\frac{\partial^3 W^3(w)}{(z-w)}\right]$$
$$+5\left[\frac{W^5(w)}{(z-w)^2} + \frac{2}{5}\frac{\partial W^5(w)}{(z-w)}\right] . \tag{2.19}$$

In terms of Fourier modes (2.19) translates into,

$$[W^3_m, W^4_n] = (3m-2n)W^5_{m+n} + q^2\frac{64}{35}[5m(m+1)(m+2) - 5(m+1)(m+2)(n+3)+$$
$$+3(n+2)(n+3)(m+2) - (n+1)(n+2)(n+3)]W^3_{m+n} \tag{2.20}$$

which agrees with (2.1) for $s = 3, s' = 4$. Similarly, one may verify that for other choices of s, s', OPEs of the fields in the tower (2.18) yield precisely the commutation relations of the PRS algebra.

In general, the OPE $W^s(z)W^{s'}(w)$ assumes the form,

$$W^s(z)W^{s'}(w) = q^{2(s-2)}\frac{2^{3s-7}s!(s-1)!(s-2)!}{(2s-3)!!}\frac{\delta_{s,s'}}{(z-w)^{s+s'}}+$$
$$+B(s)B(s')\sum_{l=1}^{s+s'-2}\frac{R_l^{ss'}(\partial\phi,\partial\bar{\phi})}{(z-w)^l} \tag{2.21}$$

where,

$$R_l^{ss'} = \sum_{k=1}^{s-1}\sum_{k'=1}^{s'-1}(-1)^{k'}A^s_k A^{s'}_{k'}[(-1)^s\frac{(s-k+k'-1)!}{(s-k+k'-l)!} : \partial^{s+k'-l}\phi\,\partial^{s'-k'}\bar{\phi} : +$$
$$+(-1)^{s'}\frac{(k+k'-1)!}{(k+k'-l)!} : \partial^{s'-k'}\phi\,\partial^{s+k'-l}\bar{\phi} :] . \tag{2.22}$$

Notice that the expression (2.22) satisfies the condition,

$$R_l^{ss'}(\partial\phi,\partial\bar{\phi}) = (-1)^{s+s'}R_l^{ss'}(\partial\bar{\phi},\partial\phi) \tag{2.23}$$

which is consistent with the Z_2 grading, $W^s \to (-1)^s W^s$ induced by the exchange of the two U(1) currents. Moreover, for any fixed value of l, the field $R_l^{ss'}$ has scaling dimension $s+s'-l$. Therefore there is a unique resolution of all $R_l^{ss'}$ with $1 \leq l \leq s+s'-2$ in terms of the conformal fields $W^{s+s'-2}$, $W^{s+s'-4}$, $\cdots$. In particular the symmetry of the problem alone guarantees the existence of numbers $\{\lambda_{l;0}^{ss'}, \lambda_{l;2}^{ss'}, \lambda_{l;4}^{ss'}, ...\}$ such that ,

$$R_{2r}^{ss'} = \lambda_{2r;0}^{ss'} W^{s+s'-2r} + \lambda_{2r;2}^{ss'} \partial^2 W^{s+s'-2r-2} + \lambda_{2r;4}^{ss'} \partial^4 W^{s+s'-2r-4} + \cdots \tag{2.24a}$$

$$R_{2r-1}^{ss'} = \lambda_{2r-1;0}^{ss'} \partial W^{s+s'-2r} + \lambda_{2r-1;2}^{ss'} \partial^3 W^{s+s'-2r-2} + \lambda_{2r-1;4}^{ss'} \partial^5 W^{s+s'-2r-4} + \cdots \tag{2.24b}$$

The series (2.24) will terminate with derivatives of W^2 if $s+s'$ is even and derivatives of W^3 if $s+s'$ is odd. Of course it is possible some of the $\lambda_l^{ss'}$ to be zero.

We may substitute (2.24) in (2.21) and compute the commutation relations of the resulting infinite dimensional algebra. We use,

$$[W_m^s, W_n^{s'}] = \oint_{C_0} \frac{dw}{2\pi i} w^{n+s'-1} \oint_{C_w} \frac{dz}{2\pi i} z^{m+s-1} W^s(z) W^{s'}(w) \tag{2.25}$$

where C_w is a contour around w and C_0 a contour around zero. The linearity of the decomposition (2.24) ensures that the relations we obtain for the commutators $[W_m^s, W_n^{s'}]$ with $s, s' \geq 2$ and $m, n \in Z$ are of the PRS type. Also, the Jacobi identity (associativity) is automatically satisfied by construction. Of course in our case, the structure constants are uniquely determined by the numerical coefficients $\lambda_l^{ss'}$. To find their values (and hence prove that the operators (2.11) provide a representation of the universal W-algebra with $c=2$), we must substitute (2.11) in (2.24) and compare with (2.22) for all l. This way we obtain linear systems of algebraic equations for the unknown coefficients $\lambda_l^{ss'}$. Results of some explicit calculations are given in the appendix. It is possible to verify directly that they yield the PRS structure constants for the commutation relations of the corresponding spin fields. However the combinatorics of the problem becomes quite complicated when s and s' are arbitrary and so far we have not been able to extract the general solution for $\lambda_l^{ss'}$ in closed form. Therefore we have to find an alternative and more systematic way to compare our OPE results with the coefficients of the PRS algebra.

At this point we recall that in 2-d CFT, the structure constants C_k^{ij} of any closed operator algebra of quasiprimary fields,

$$A^i(z) A^j(w) = \sum_k C^{ij}{}_k (z-w)^{\Delta_k - \Delta_i - \Delta_j} [A^k(w) + derivatives] \tag{2.26}$$

are determined unambiguously using the 3-point functions, [6]

$$\langle A^i(z_1) A^j(z_2) A^k(z_3) \rangle = \frac{\Gamma^{ijk}}{z_{12}^{\Delta_{ij}} z_{13}^{\Delta_{ik}} z_{23}^{\Delta_{jk}}} \tag{2.27}$$

by,

$$\Gamma^{ijk} = \sum_{k'} C_{k'}^{ij} G^{k'k} \tag{2.28}$$

where,

$$\langle A^i(z_1)A^j(z_2)\rangle = \frac{G^{ij}}{z_{12}^{\Delta_i+\Delta_j}}\delta_{\Delta_i,\Delta_j} \tag{2.29}$$

and $z_{ij} = z_i - z_j$, $\Delta_{ij} = \Delta_i + \Delta_j - \Delta_k$, etc. In our case, conformal invariance implies that

$$\langle W^{s_1}(z_1)W^{s_2}(z_2)W^{s_3}(z_3)\rangle = \frac{1+(-1)^{s_1+s_2+s_3}}{z_{12}^{s_{12}}z_{13}^{s_{13}}z_{23}^{s_{23}}}B(s_1)B(s_2)B(s_3)K_{s_3}^{s_1s_2} \tag{2.30}$$

which vanishes for odd values of $s_1 + s_2 + s_3$, consistent with the Z_2 grading of the W-fields (2.11) under the interchange $\phi \leftrightarrow \bar{\phi}$. Using Wick's theorem for the free-boson representation of the W-generators we have obtained all 3-point functions (2.30) in closed form. The coefficients $K_{s_3}^{s_1s_2}$ turn out to be,

$$K_{s_3}^{s_1s_2} = (-1)^{s_1+s_2}\sum_{n=1}^{s_1-1}\sum_{m=1}^{s_2-1}\sum_{k=1}^{s_3-1}\frac{(-1)^{n+k}A_n^{s_1}A_m^{s_2}A_k^{s_3}}{z_{12}^{s_3-s_2+m-n}z_{13}^{s_2-s_1+n-k}z_{23}^{s_1-s_3+k-m}}\cdot$$

$$\cdot(s_1+m-n-1)!(s_2+k-m-1)!(s_3+n-k-1)!\ . \tag{2.31}$$

In (2.30) we have displayed explicitly the correct singularity structure of the correlation functions. Therefore conformal invariance guarantees that the prefactor $K_{s_3}^{s_1s_2}$ is in fact independent of z_1, z_2, z_3. Although this is not obvious by simple inspection of (2.31) one may find amusing (?) to verify this statement explicitly for different values of s_i. For this reason we may choose $z_1 = 1$, $z_2 = 0$, $z_3 = -1$ in order to evaluate $K_{s_3}^{s_1s_2}$. Setting $s_1 = s$, $s_2 = s'$ we find that in the special case $s_3 = s + s' - 2$,

$$K_{s+s'-2}^{ss'} = \frac{2}{ss'}\binom{2s-3}{s-1}\binom{2s'-3}{s'-1}(s+s'-2)!(s+s'-4)!\ . \tag{2.32}$$

This identity is highly nontrivial and so far we have not been able to prove it analytically. However, we checked its validity thoroughly using numerical methods. Putting everything together we obtain

$$\langle W^s(z_1)W^{s'}(z_2)W^{s+s'-2}(z_3)\rangle = (s+s'-2)\frac{c_{s+s'-2}}{z_{12}^2z_{13}^{2(s-1)}z_{23}^{2(s'-1)}} \tag{2.33}$$

where,

$$c_{s+s'-2} = q^{2(s+s'-4)}2^{3s+3s'-13}\frac{(s+s'-2)!(s+s'-3)!(s+s'-4)!}{(2s+2s'-7)!!} \tag{2.34}$$

is the coefficient of the 2-point function of $W^{s+s'-2}$, (c.f. (2.21)). Taking into account the normalizations (2.27-2.29) we conclude that

$$W^s(z)W^{s'}(w) = (s+s'-2)\frac{W^{s+s'-2}(w)}{(z-w)^2} + \cdots \tag{2.35}$$

for $s, s' \geq 2$. In the notation introduced in (2.24) this is equivalent to

$$B(s)B(s')\lambda_{2;0}^{ss'} = s + s' - 2\ . \tag{2.36}$$

Conformal invariance fixes automatically the OPE coefficients of the derivative terms as well. In general let A_i, $i = 1, 2, 3$ be quasiprimary operators and

$$A^1(z)A^2(w) = C_3^{12}\ (z-w)^{-\Delta_{12}}[A^3(w) + \sum_{n=1}^{\infty} \kappa_n (z-w)^n \partial_w^n A^3(w)]\ . \tag{2.37}$$

We can compute the 3-point function $\langle A^1(z_1)A^2(z_2)A^3(z_3)\rangle$ in two ways. First we know that it must have the form (2.27). Second we can compute it using the OPE (2.37). Matching the coefficients of the Laurent expansion in z_{12} of the two expressions for the 3-point function we obtain

$$\kappa_n = \frac{\Delta_{13}}{2\Delta_3}\frac{(\Delta_{13}+1)}{(2\Delta_3+1)}\cdots\frac{(\Delta_{13}+n-1)}{(2\Delta_3+n-1)} \equiv \frac{(\Delta_{13})_n}{(2\Delta_3)_n}\ . \tag{2.38}$$

In our case, (2.38) fixes the coefficient of $\partial W^{s+s'-2}$,

$$B(s)B(s')\lambda_{1;0}^{ss'} = s - 1 \tag{2.39}$$

which together with (2.36) generate the commutation relations (1.3) that describe the leading (highest spin) structure of the PRS universal W-algebra.

Similarly we may compute the coefficient of any other term in the OPE (2.21) by evaluating $K_{s''}^{ss'}$ for $s'' = s+s'-4,\ s+s'-6, \cdots$. Howeverm, we have no general analytic proof that all the remaining (subleading) structure constants of our algebra coincide with that of PRS for arbitrary s, s'. Nevertheless we have verified it extensively using symbolic manipulation programs.

The advantage of using 3-point functions to determine the coefficients $\lambda_l^{ss'}$ relies on the fact that we can obtain closed expressions for the 3-point functions in terms of triple sums (c.f. (2.30),(2.31)), which we can then apply to any specific case separately.

3. **Highest weight representations**

It is obvious from the preceding that the W_∞ algebra we constructed is a subalgebra in the enveloping algebra of the $U(1) \otimes U(1)$ current algebra generated by $i\partial\phi$, $i\partial\bar{\phi}$. The W-generators are bilinears in the current modes. Thus, highest weight (hw) representations of the $U(1) \otimes U(1)$ algebra decompose into representations of the PRS algebra with $c = 2$. In fact, a very simple and wide class of hw representations of the PRS algebra are furnished by hw representations of the $U(1) \otimes U(1)$ current algebra.

The standard $SL(2)$ invariant vacuum $|0\rangle$ is also invariant under the PRS algebra,

$$W_n^s|0\rangle = 0\ \ ,\ \ n \geq 1 - s\ . \tag{3.1}$$

A state $|Q\rangle$ will be a hw state of the PRS algebra if it satisfies,

$$W_{n>0}^s|Q\rangle = 0\ \ ,\ \ W_0^s|Q\rangle = Q_s|Q\rangle \tag{3.2}$$

where, in particular, $\Delta \equiv Q_2$ is the (left) scaling dimension. It is obvious that the standard hermiticity condition of the $U(1)\otimes U(1)$ current algebra translates into $(W^s_n)^\dagger = (-1)^s W^s_{-n}$ for the PRS algebra. Also, in analogy with the hw representations of the Virasoro algebra, the states $|Q\rangle$ can be created from the vacuum by local operators $V_Q(z)$, $V_Q(0)|0\rangle = |Q\rangle$. Then, the hw conditions (3.2), translate into the following OPE,

$$W^s(z)V_Q(w) = Q_s \frac{V_Q(w)}{(z-w)^s} + O[(z-w)^{1-s}] \ . \tag{3.3}$$

It is easy to see that the vertex operators,

$$V_{a,\bar{a}}(z) =: exp(ia\phi + i\bar{a}\bar{\phi}) : \tag{3.4}$$

are primary (hw) operators of the PRS algebra with charges,

$$Q^{a,\bar{a}}_s = (-1)^s q^{s-2} \frac{2^{s-3}[(s-2)!]^2}{(2s-3)!!}[1+(-1)^s]\Delta \ , \ 2\Delta = |a|^2 \ . \tag{3.5}$$

These operators have zero charges under the odd W-generators. This does not mean, however, that the odd part of the algebra acts trivially on the representation module. In fact we can prove that the vertex operators (3.4) exhaust all hw representations of the PRS algebra with $c = 2$. The proof relies on the connection of the construction given above to the theory of Z_∞ parafermions and will be dealt with in the next section. Also, as we will see later, this automatically yields the (reduced) character formulae for hw irreducible unitary representations of the PRS algebra with $c = 2$,

$$\chi^{\bar{a}}_a(q) \equiv Tr[q^{L_0 - \frac{c}{24}}] = \frac{q^{\frac{|a|^2}{2} - \frac{1}{12}}}{\prod_{n=1}^{\infty}(1-q^n)^2} \ . \tag{3.6}$$

Here, $q = \exp(2\pi i\tau)$ depends on the modular parameter τ and should be distinguished from the PRS parameter q appearing in (3.5). Equation (3.6) implies in particular that an affine irreducible $U(1)\otimes U(1)$ representation decomposes into (and in fact coincides with) exactly one W_∞ irreducible representation.

The vertex operators (3.4) provide the first concrete class of hw, *unitary* representations of the W_∞ algebra.

4. Z_∞ parafermions

The construction described in the previous section is inspired and closely related to the theory of Z_∞ parafermions. The Z_N parafermion algebra can be described in two complementary ways, either as the chiral algebra of the coset $SU(N)_1 \otimes SU(N)_1/SU(N)_2$, (c.f. (1.2)), or as that of the coset $SU(2)_N/U(1)$. We will pursue the second approach in order to avoid unnecessary complications dealing with the large N limit of $SU(N)$.

The Z_N parafermions can be defined using the $SU(2)$ level-N current algebra, [7],

$$J^a(z)J^b(w) = \frac{N}{2}\frac{\delta^{ab}}{(z-w)^2} + i\varepsilon^{abc}\frac{J^c(w)}{z-w} \ . \tag{4.1}$$

In particular, we set, [8]

$$J^3 = i\sqrt{N/2}\,\partial\rho\,,\ J^{\pm} = \sqrt{N}exp(\pm\sqrt{\frac{2}{N}}i\rho)\,\psi_{\pm 1}\ ,\ \langle\rho(z)\rho(w)\rangle = -\log(z-w) \tag{4.2}$$

and the OPE of the parafermions ψ_k assumes the form

$$\psi_k(z)\psi_{k'}(w) = c_{k,k'}(z-w)^{-\frac{2kk'}{N}}\left(\psi_{k+k'}(w) + O(z-w)\right)\ , \tag{4.3a}$$

$$\psi_k(z)\psi_{-k}(w) = (z-w)^{-2\frac{k(N-k)}{N}}[1 + \frac{k(N-k)(N+2)}{N(N-1)}(z-w)^2T_\psi(w) + ...]\ , \tag{4.3b}$$

where,

$$\psi_k = \psi^\dagger_{-k}\ \ ,\ \ \psi_{N+k} = \psi_k\ ,\ \psi_0 = 1 \tag{4.3c}$$

and T_ψ is the parafermionic stress tensor. We point out that at $N=\infty$, $SU(2)$ "flattens" and becomes a $U(1)^3$ current algebra. This follows by simple rescaling of the $SU(2)$ currents in (4.1) as $J^a(z)\to N^{-1/2}J^a(z)$. Then, the large (level) N limit of the $SU(2)$ current algebra is well defined and involves no infinities. Considering the coset $SU(2)/U(1)$ means removing the scalar field ρ from the spectrum.

The spin of the parafermions ψ_k is given by $\Delta_k = k(N-k)/N$ for $k=0,1,2,\cdots,N-1$. As $N\to\infty$, the scaling dimension of $\psi_{\pm k}$, $\Delta_{\pm k} = k(1-\frac{k}{N})\to k$. Thus $N=2$ and $N=\infty$ are the only two cases where the parafermionic algebra becomes local. For $N=\infty$, the discrete $Z_N\otimes Z_N$ symmetry of the system is promoted to affine $U(1)\otimes U(1)$. In this limit, strictly speaking, the discrete symmetry becomes a global continuous chiral symmetry. However in 2-d statistical models at criticality, such a symmetry becomes automatically local and the $U(1)\otimes U(1)$ currents can be identified with $\psi_1 = i\partial\phi$ and $\psi_1^\dagger = i\partial\bar{\phi}$. Also, $\psi_k\sim(\partial\phi)^k$, $\psi_k^\dagger\sim(\partial\bar{\phi})^k$ are expressed as composite fields for $k\geq 2$.

The (integer spin) chiral algebra of the Z_N theory contains (for all $2\leq s\leq N$) a unique field of spin s which emerges on the right hand side of (4.3b). Then, according to (4.3b) the W-generators appear in the non-singular terms of the operator product of $\partial\phi$ and $\partial\bar{\phi}$ and therefore must be of the form advocated in (2.11). It is quite standard to consider as the symmetry algebra of the Z_N parafermion theory either the algebra (4.3) (with fractional spins) or equivalently the W_N which is formed out of "composite" parafermion fields. For each parafermionic primary field there corresponds a finite number of primary fields of the W_N algebra, [8,9].

As the level N of the $SU(2)$ current algebra (4.1) becomes large, the Hilbert space of the parafermion theory decomposes into three interesting classes of primary fields. The first class, denoted by C_0, encompasses the fields whose dimensions approach zero.

The second, C_F, contains the fields that reach finite scaling dimensions, while the third, C_∞, contains the operators that have unbounded dimensions as $N \to \infty$. The class C_∞ deserves a study in its own right and we suspect that there might be non-trivial structure hidden in it. Here, we mainly deal with C_F and we will comment briefly on C_0. For simplicity, as we explained earlier, it is more convenient to consider the large (level) N limit of the SU(2) WZW model first and then translate the results to the parafermionic coset (1.2) with $p = 1$.

Let $[j]$ be a hw representation of SU(2) with highest spin j, which contains $2j + 1$ states labeled by m, with $-j \leq m \leq j$. All such $2j+1$ states are hw with respect to W_N and the $m = j$ state corresponds to the parafermionic primary field, [8,9]. Also the affine $SU(2)$ primary fields are labeled by their spin, $j = 0, \frac{1}{2}, 1, ...$ and the third component m, $-j \leq m \leq j$. It is known that their dimension is given by $\Delta_j = \frac{j(j+1)}{N+2}$. In analogy with the parafermionic case, we have three sectors of fields as $N \to \infty$. $\tilde{C}_0$ contains primary fields at finite (fixed) j with weight $\Delta_j \sim \frac{1}{N}$, $\tilde{C}_F$ contains spins $j \sim \sqrt{N}$ and $\tilde{C}_\infty$ contains spins $j \sim N$. The fields in $\tilde{C}_0$ form a closed operator algebra. This statement turns out to be trivial because the OPEs of the $\tilde{C}_0$ representations follow exactly the group theory. In particular, the OPE coefficients calculated in [10] reduce to the SU(2) Glebsch-Gordan coefficients. For completeness we point out that the statement we made above for $\tilde{C}_0$ is also true for any collection of fields with $\Delta \to 0$, due to dimensional arguments.

For the fields in $\tilde{C}_F$ the situation is more interesting. Let us parametrize them using their dimension $\Delta = \lim_{\substack{j\to\infty \\ N\to\infty}} \frac{j(j+1)}{N+2}$, which is finite provided that $j/\sqrt{N}$ is held fixed. (It is obvious that any real value for Δ can be obtained this way.) Now, if we analyze the OPE coefficients in this regime we will find that the (3-point) coupling $C(\Delta_1, \Delta_2, \Delta_3)$ is exponentially suppressed as $e^{-\beta\sqrt{N}}$, unless one of the following relations holds

$$\sqrt{\Delta_1} = \sqrt{\Delta_2} + \sqrt{\Delta_3} \tag{4.4}$$

or cyclic permutations. This is precisely the composition law for abelian vertex operators! Since the $SU(2)$ currents in the large N limit become abelian, the primary fields in the $\tilde{C}_F$ sector must be of the form $: exp[ia_3\rho + ia\phi + i\bar{a}\bar{\phi}] :$, where ρ was introduced in (4.2) and

$$i\partial\phi(z) \equiv \lim_{N\to\infty} N^{-\frac{1}{2}} J^+(z) = \psi_1(z) \ , \ i\partial\bar{\phi}(z) \equiv \lim_{N\to\infty} N^{-\frac{1}{2}} J^-(z) = \psi_{-1}(z) \ . \tag{4.5}$$

The parameter a_3 is proportional to the eigenvalue of the zero mode J^3_0. In this language, the W_∞ primary fields can be obtained by factoring out the ρ-field dependence and therefore they assume the form (2.11). The rest of the representation is obtained in the standard way by acting with the lowering operators of the U(1) currents $\partial\phi$ and $\partial\bar{\phi}$. This construction proves that the representations of the PRS W_∞ algebra are isomorphic to the representations of the $U(1) \otimes U(1)$ current algebra, as we alluded to in the previous section.

Next, we analyze further the structure of these representations by deriving their character formulae. Recall that the character of an affine SU(2) representation with spin j is given by, [9]

$$\chi_j(\tau,z) \equiv Tr_j[q^{L_0-\frac{c}{24}} e^{2\pi i J_0^3}] = i\frac{\vartheta_{2j+1,N+2}(z|\tau)-\vartheta_{-2j-1,N+2}(z|\tau)}{\vartheta_1(z|\tau)} \tag{4.6}$$

where,

$$\vartheta_{m,N}(z|\tau) = \sum_{n\in Z+m/2N} e^{2\pi i N(n^2\tau - nz)} \tag{4.7}$$

and $\vartheta_1(z|\tau)$ is the standard ϑ-function. The reduced character, is obtained from (4.6) by setting $z=0$ and provides the building blocks of the SU(2) WZW partition function in the absence of background fields. Then, the corresponding characters of the hw representations of the W_N algebra are given in terms of the string functions c^l_m, with $l=2j$,

$$\chi_l(\tau,z) = \sum_{m=1-k}^{k} c^l_m(\tau)\vartheta_{m,k}(z|\tau)\ . \tag{4.8}$$

The characters are $\chi^l_m = \eta c^l_m$, where η is the Dedekind η-function. Now by taking the $N\to\infty$ limit for the class of $\tilde{C}_F$ representations we find,

$$\chi_j(\tau) = \sqrt{\frac{\Delta}{N}}\frac{e^{2\pi i\tau\Delta}}{\eta^3} + O(\frac{1}{N}) \tag{4.9}$$

which means that the Z_∞ parafermionic characters are given by (3.6), as advertized.

We should mention at this point that there is a RW_N algebra which constitutes the real counterpart of the complex W_N algebra studied in this paper. The algebra we have already studied is obtained from the chiral algebra of the coset model $SU(N)_1\otimes SU(N)_1/SU(N)_2$ and has $c=2$ in the limit $N\to\infty$. On the other hand, the real coset $O(N)_1\otimes O(N)_1/O(N)_2$ has $c=1$ (for all N) and can be identified with the sequence of Z_2 orbifold models with radius $R=N\sqrt{2}$. It is known, [11], that these models have a chiral algebra generated by the stress tensor, a spin-4 field (which is quartic in the U(1) current) and a field of spin N which is represented by the vertex operator $:cos(\sqrt{2N}\phi):$. The rest of the fields with spin $6,8,\cdots$ appear in the OPE of the basic fields with spin 2,4,N. As $N\to\infty$, the field $:cos(\sqrt{2N}\phi):$ decouples and we are left with the RW_∞ algebra generated by fields with even spin only. It can be shown that these fields are given (up to normalization) by,

$$W^{2s}(z) \sim \sum_{k=1}^{2s-1}(-1)^k A^{2s}_k \partial^k\phi\partial^{2s-k}\phi \tag{4.10}$$

where A^{2s}_k has been defined in (2.15) and ϕ is a single real scalar field. Hence, RW_∞ is the quotient of the W_∞ by the ideal of fields with odd spins.

5. **Conclusions and future directions**

In this paper we presented a realization of the PRS W_∞ algebra with $c = 2$ in terms of a free complex boson. In our construction associativity is manifest. We also gave a class of irreducible hw representations of this algebra in terms of vertex operators. To our knowledge this is the first construction of highest weight representations for W_∞. The whole line of thought was motivated by the large N limit of Z_N parafermions. We presented an analysis of the structure of the underlying W-algebra together with its parafermionic Hilbert space, when N becomes large. In particular we noticed that the hw representations of the W_∞ algebra we constructed emerge naturally as the limits of hw representations of the W_N algebra. This enabled us to prove that they are unitary and irreducible and to give also a character formula. However we neglected in our discussion some subsets of the Hilbert space that were not relevant (at least directly) to the construction we presented. It might be worthwhile to study further the C_∞ class of fields and analyze its structure. Also, it would be interesting to extend our results for values of central charge $c > 2$.

The need to investigate the structure of W_N algebras as $N \to \infty$ arises not only in problems of 2-d CFT but also in other areas of mathematical physics. Some (rather curious) connections of W_∞ with the infinite dimensional (hidden) symmetry algebra of Euclidean self-dual 4-d spaces (gravitational instantons) have already been discussed in the literature and we refer the reader to [3,12] for further details. (See also [13,14] for a complementary point of view on some related issues of hyper-Kahler geometry). Here we only intend to comment briefly on the relevance that large N limits of W algebras seem to have in the theory of "universal" systems of integrable non-linear differential equations.

Let us consider the KP hierarchy of non-linear differential equations (see for instance [15] and references therein) described by the evolution,

$$\frac{\partial Q}{\partial t_r} = [(Q^r)_+, Q] \tag{5.1}$$

where Q is a formal pseudo-differential operator,

$$Q = \partial_z + q_1(z, t_i)\partial_z^{-1} + q_2(z, t_i)\partial_z^{-2} + \cdots \tag{5.2}$$

and $(Q^r)_+$ denotes the purely differential part of Q^r. Then for every fixed positive integer, r, we obtain a system of infinitely many coupled non-linear equations of the form,

$$\frac{\partial q_i}{\partial t_r} = F_i^{(r)}(q, q', q'', \cdots) \ , \ \ i = 1, 2, \cdots \tag{5.3}$$

where $F_i^{(r)}$ are certain differential polynomials in the q-variables, determined by (5.1). It is known that all KdV systems of $SL(N)$-type hierarchies (whose members are parametrized by r) are special cases of (5.1). For each value of r, their embedding in the KP hierarchy is described simply by the requirement that Q is the (unique) N-th root of a differential operator L_N,

$$Q^N \equiv L_N = \partial_z^N + u_{N-2}(z)\partial_z^{N-2} + \cdots + u_0(z) \ . \tag{5.4}$$

Then, the equations (5.3) reduce to a system of $N-1$ independent differential equations and the rest are functionally related to them.

At this point recall that the Hamiltonian structure of the KdV systems can be formulated using the commutation relations of the Gelfand-Dickey algebra $GD(SL(N))$ (see for instance [16] and references therein). For $N = 2$ this reduces to the standard description of the KdV equation in terms of the Virasoro algebra, [17]. On the other hand, it has been established that Gelfand-Dickey algebras provide a classical Hamiltonian framework for all W_N symmetry algebras with $N \geq 2$ (see [18] and references therein). Therefore it is natural to view the full unrestricted KP hierarchy as a universal KdV system associated with a large N limit of W_N algebras. This observation alone suggests an interesting reformulation of the problem (5.1) and its integrability properties¶. We suspect that there is a deeper connection among the KP hierarchy, the algebra of area preserving diffeomorphisms (1.3) and its unitary representations studied in this paper. We also think that a better understanding of these issues will hopefully clarify the different realizations of W_N in the limit $N \rightarrow \infty$. Work in this direction is in progress.

Large N limits also became popular recently in the context of Universal Yang-Mills theory [19] and non-perturbative 2-d quantum gravity [20], where certain structures of the type described above seem to emerge. It is highly plausible that there exists a unified formulation of various large N limits in terms of the Hilbert space theory of factors. Subsequent specification of a Hamiltonian will generate a KP-type hierarchy. Such issues beg for a better understanding and will be considered elsewhere in more detail.

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Note Added

¶D. Fairlie and L. Dickey have raised this point independently. It is also transparent from a footnote in Morozov's paper, [4], that A. Radul has been working on similar ideas.

After this work was complete, we became aware of reference [21], where the idea to use $U(1)$ affine algebras in order to realize W_∞ was presented independently (but rather abstractly). In contrast, the results we describe here are explicit and quite natural in the context of 2-d CFTs. It might be interesting to generalize our field-theoretic representation to the lone star W_∞ algebra, also introduced in [5].

Appendix

We summarize the results of the OPE $W^s(z)W^{s'}(w)$ for (s,s') =(2,2), (2,3), (2,4), (2,5), (2,6), (3,3), (3,4), (4,4). The numerical coefficients $\lambda_l^{ss'}$ that we omit in these cases are identically zero. We have,

$$\lambda^{22}_{1;0} = 1 \ , \ \lambda^{22}_{2;0} = 2 \ , \ \lambda^{23}_{1;0} = \frac{1}{2q} \ , \ \lambda^{23}_{2;0} = \frac{3}{2q} \ , \ \lambda^{24}_{1;0} = \frac{5}{16q^2} \ , \ \lambda^{24}_{2;0} = \frac{5}{4q^2}$$

$$\lambda^{24}_{4;0} = 12 \ , \ \lambda^{25}_{1;0} = \frac{7}{32q^3} \ , \ \lambda^{25}_{2;0} = \frac{35}{32q^3} \ , \ \lambda^{25}_{4;0} = \frac{30}{q} \ , \ \lambda^{26}_{1;0} = \frac{21}{128q^4}$$

$$\lambda^{26}_{2;0} = \frac{63}{64q^4} \ , \ \lambda^{26}_{4;0} = \frac{105}{2q^2} \ , \ \lambda^{26}_{4;2} = 12 \ , \ \lambda^{26}_{5;0} = -120 \ , \ \lambda^{26}_{6;0} = 240$$

$$\lambda^{33}_{1;0} = \frac{1}{2q^2} \ , \ \lambda^{33}_{1;2} = \frac{2}{5} \ , \ \lambda^{33}_{2;0} = \frac{1}{q^2} \ , \ \lambda^{33}_{2;2} = \frac{9}{5} \ , \ \lambda^{33}_{3;0} = 6$$

$$\lambda^{33}_{4;0} = 12 \ , \ \lambda^{34}_{1;0} = \frac{5}{16q^3} \ , \ \lambda^{34}_{1;2} = \frac{2}{7q} \ , \ \lambda^{34}_{2;0} = \frac{25}{32q^3} \ , \ \lambda^{34}_{2;2} = \frac{12}{7q}$$

$$\lambda^{34}_{3;0} = \frac{8}{q} \ , \ \lambda^{34}_{4;0} = \frac{24}{q} \ , \ \lambda^{44}_{1;0} = \frac{75}{256q^4} \ , \ \lambda^{44}_{1;2} = \frac{15}{16q^2} \ , \ \lambda^{44}_{1;4} = \frac{3}{14}$$

$$\lambda^{44}_{2;0} = \frac{75}{128q^4} \ , \ \lambda^{44}_{2;2} = \frac{75}{16q^4} \ , \ \lambda^{44}_{2;4} = \frac{10}{7} \ , \ \lambda^{44}_{3;0} = \frac{135}{8q^2} \ , \ \lambda^{44}_{3;2} = 8$$

$$\lambda^{44}_{4;0} = \frac{135}{4q^2} \ , \ \lambda^{44}_{4;2} = 36 \ , \ \lambda^{44}_{5;0} = 120 \ , \ \lambda^{44}_{6;0} = 240$$

Notice that in (2.21), $B(s)B(s')$ appears as a prefactor.

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