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On Null States in Sub-Critical Strings

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Abstract

It is shown that in a generic theory of sub-critical strings (or alternatively, two-dimensional gravity coupled to matter) the physical states are in one to one correspondence with the primary fields of the matter (critical) theory.

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There has been considerable progress recently on sub-critical string theories following Polyakov's treatment of two-dimensional gravity in the light-cone gauge, [1] and subsequent understanding of his results in the conformal gauge, [2]. In parallel, random matrix models, [3], have be used with success in order to formulate non-perturbative two-dimensional gravity coupled to matter, [4,5,6].

The meaning of the non-perturbative formulation of two-dimensional gravity is obscure in the continuum approach. It is not even possible to solve the continuum theory at fixed genus let alone sum over geni. It seems imperative to pursue our understanding of the continuum theory since it may turn out to be better in dealing with certain issues concerning among others the appearance and meaning of hidden symmetries. In this note we intend to characterize explicitly the physical states of two-dimensional gravity coupled to matter, in the continuum formulation.

When we couple a Conformal Field Theory (CFT) to two-dimensional gravity, after gauge fixing in the conformal gauge, one ends up with three sectors, the matter theory itself (with central charge d), the b-c ghost system (with central charge -26) and Liouville theory which adjusts its central charge to 26-d. The physical states of the theory are the states in the tensor product of the three separate Hilbert spaces, which are annihilated by the BRST charge. These are in one to one correspondence with states in the Liouville + matter theory which are primary with respect to the combined Liouville + matter stress tensor (with central charge c=26) and have dimension one, [7].

In order to find the physical states let's start from a specific irreducible representation of the matter theory generated by the primary state $|\Delta\rangle_M$. To find a physical state we tensor the previous representation with an irreducible representation in the Liouville sector generated by the primary state $|1-\Delta-N\rangle_L$. The tensor product decomposes generically into an infinite number of irreducible representations with respect to $T_{ML}=T_M+T_L$, where T_M is the stress tensor of the matter theory and T_L the stress tensor of Liouville theory.

In particular there will be $p(N)$ primary states (with respect to T_{ML}) with dimension one (and thus physical) in the tensor product above, where $p(N)$ is defined as usual,

$$\prod_{n=1} \frac{1}{(1-q^n)} = \sum_{n=1}^{\infty} p(n) q^n \tag{1}$$

Thus it seems that for each irreducible representation of the matter theory there will be an infinite number for physical states in the combined matter+gravity theory. What we will show in this note is that the counting above is too naive. We will prove that, despite appearances, for each irreducible representation of the matter theory there corresponds a *single* physical state of the combined system which in fact is[‡] $|\Delta\rangle_M \otimes |1-\Delta\rangle_L$. This has been conjectured in [4]. It is a generalization of a similar thing happening in critical strings where the role of matter is played by the 25 bosons with positive signature and that of Liouville by the 1 boson of negative signature, ([8] and references therein).

Let us describe some simple examples that will demonstrate what is really happening. Consider the tensor product of two Virasoro representations generated by the highest weight vectors (hwv's), $|\Delta_1\rangle_M$, $|\Delta_2\rangle_L$. Let us find at level 1, a primary state with respect to the combined stress tensor, T_M+T_L. There are two states at level one. One of them will be the descendant of the hwv, $[L_{-1}^M + L_{-1}^L]|\Delta_1\rangle_M \otimes |\Delta_2\rangle_L$ and the other one, which is primary, is given by,

$$|\chi_1\rangle = [L_{-1}^M - \frac{\Delta_1}{\Delta_2} L_{-1}^L]|\Delta_1\rangle_M \otimes |\Delta_2\rangle_L \tag{2}$$

It is easy to calculate the norm of this state which is equal to $2\Delta_1(\Delta_1 + \Delta_2)/\Delta_2$. In order for such a state to be physical its dimension must be one, that is $\Delta_1 + \Delta_2 + 1 = 1$. Thus, it is obvious that when this state is physical, it is also null. At level two there are five independent states, two out of them

[‡]An index M (L) on a state means that the state is primary with respect to the matter (Liouville) stress tensor. A index ML implies that the state is primary with respect to the combined stress tensor, $T_M + T_L$.

being primary. The most general state is of the form,

$$[L^M_{-2} + \alpha_1 (L^M_{-1})^2 + \alpha_2 L^M_{-1} L^L_{-1} + \alpha_3 L^L_{-2} + \alpha_4 (L^L_{-1})^2] |\Delta_1\rangle_M \otimes |\Delta_2\rangle_L \qquad (3)$$

Imposing the condition, that $L^M_{m>0} + L^L_{m>0}$ vanish on it we obtain

$$3 + \alpha_1 (4\Delta_1 + 2) + 2\alpha_2 \Delta_2 = 0 \ , \ 2\Delta_1 \alpha_2 + 3\alpha_3 + \alpha_4 (4\Delta_2 + 2) = 0 \qquad (4a)$$

$$4\Delta_1 + \frac{d}{2} + 6\alpha_1 \Delta_1 + \alpha_3 (4\Delta_2 + 13 - \frac{d}{2}) + 6\alpha_4 \Delta_2 = 0 \qquad (4b)$$

There is only one undetermined parameter left in (4) consistent with the fact that there are two independent primary states at level two. A straightforward computation of the norm of the general linear combination of the two primary states shows that it is proportional to $\Delta_1 + \Delta_2 + 1$. Again, when these states become physical, their norm vanishes. When these states have dimension one, ($\Delta_2 = -\Delta_1 - 1$) we can write them in the following suggestive form,

$$|\chi_2\rangle = L^{ML}_{-1} (L^M_{-1} + \frac{\Delta_1}{\Delta_1 + 1} L^L_{-1}) |\Delta_1\rangle \otimes | - \Delta_1 - 1\rangle \qquad (5a)$$

$$|\psi_2\rangle = L^{ML}_{-2} + \frac{3}{2} (L^{ML}_{-1})^2 |\Delta_1\rangle \otimes | - \Delta_1 - 1\rangle \qquad (5b)$$

$$\langle\chi_2|\chi_2\rangle = \langle\psi_2|\psi_2\rangle = \langle\psi_2|\chi_2\rangle = 0 \qquad (5c)$$

.

We will digress at this point to discuss the structure of degenerate representations of the Virasoro algebra with $c = 26$ and integer dimension. The zeros of the Kač determinant at $c = 26$ are at $\Delta = -[(2r + 3s)^2 - 25]/24$ where r, s are positive integers, [9]. In particular in fig. 1 we show the embedding pattern of representations relevant for our case. An arrow pointing from one hwv to another means that the latter is a null hwv constructed on the Verma module of the former. The diagram is commutative, that is null hwv's constructed in a sequence along different paths on the diagram are the same. We can infer from fig. 1 that the hwv with $\Delta = 0$ has a null hwv with $\Delta = 1$. This is precisely the state $|\chi_1\rangle$ we found in (2). In the same way, $\Delta = -1$ has a null hwv at level two with $\Delta = 1$.

Let us now show how we can explain our findings above, in terms of the diagram of fig. 1. In order to obtain a physical state (hwv with $\Delta = 1$) at level one we had to start from $|\Delta\rangle_M \otimes |-\Delta\rangle_L$. At level one we are expecting a single hwv. However the diagram in fig. 1 tells us that we can construct a null hwv at level one on the $|\Delta\rangle_M \otimes |-\Delta\rangle_L \equiv |0\rangle_{ML}$, hwv. Thus this is the hwv state at level one we were looking for and moreover it is null. Continuing, in order to look for physical states at level two we must start from $|\Delta\rangle_M \otimes |-\Delta-1\rangle_L$. This contains a hwv at level one and two hwv at level two. The original hwv, $|-1\rangle_{ML}$ has $\Delta = -1$ and thus has a null hwv at level two. The hwv at level one has $\Delta = 0$ and thus has a null hwv at level one $\Delta = 1$ which is linearly independent from the previous null state, (as obvious from the figure). Thus we already constructed two null hwv's at level two, as many as expected, proving that all of them are in fact null. The pattern above generalizes with minor modifications (having to do with double counting) and in order to prove our assertion we will have to count carefully. It will turn out that the combinatorics of the problem is equivalent to Euler's pentagonal identity.

Thus our strategy is as follows. Starting from $|\Delta\rangle_M \otimes |-\Delta-N+1\rangle_L$ we decompose the tensor product of the two representations under the combined Virasoro as follows,

$$|\Delta\rangle_M \otimes |-\Delta-N+1\rangle_L \sim |-N+1\rangle_{ML}^{p(0)} \oplus |-N+2\rangle_{ML}^{p(1)} \oplus \cdots \oplus |0\rangle_{ML}^{p(N-1)} \oplus |1\rangle_{ML}^{p(N)} \oplus \cdots \tag{6}$$

where the superscripts $p(i)$ (defined in (1)) on top of a representation indicate the number of linearly independent hwv's at level i. In particular there will be $p(N)$ physical states in the tensor product. Using the diagram of fig. 1 we will construct a certain number of null hwv's at level N and then show that this number is equal to $p(N)$, proving thus, that all the would be physical states are null.

Before stating the general counting let us work another example where overlaps come into play. Consider the case the case $N = 7$. The decomposition (6) takes the form,

$$|\Delta\rangle_M \otimes |-\Delta-6\rangle_L \sim |-6\rangle_{ML}^{p(0)} \cdots \oplus |-4\rangle_{ML}^{p(2)} \cdots \oplus |-1\rangle_{ML}^{p(5)} \oplus |0\rangle_{ML}^{p(6)} \oplus |1\rangle_{ML}^{p(7)} \oplus \cdots \tag{7}$$

where only hwv's that are useful for constructing null vectors have be portrayed. ¿From the embedding diagram we see that $|-6\rangle$ has one null hwv at level 5 and another one at level 6. Also $|-4\rangle$ has one null hwv at level 3 and another one at level 4. Moreover these are linearly independent from the former (otherwise there would be an arrow connecting $|-6\rangle$ to $|-4\rangle$ in the diagram). Thus out of a total of $p(5)$ hwv's with $\Delta = -1$ $p(0)+p(2)$ of them are null and the rest $p(5)-p(0)-p(2)$ have non-zero norm. In the same way out of a total $p(6)$ hwv's with $\Delta = -1$, $p(0)+p(2)$ of them are null and the rest $p(6)-p(0)-p(2)$ have non-vanishing norm. Now $|-1\rangle$ has a null hw descendant at level two, and $|0\rangle$ has one at level one. Thus each of the hwv with $\Delta = 0, -1$ will generate one null hwv with $\Delta = 1$, a total of $p(6)+p(5)-2p(0)-2p(2)$ such states. However, not all the remaining hwv's with $\Delta = 0, -1$ which are already null will generate independent null states at $\Delta = 1$. In fact due to the commutativity of the diagram in fig. 1 only half of them are independent, there number being $\frac{1}{2}[(p(0)+p(2))+(p(0)+p(2))]$. Thus the total number of null hwv's with $\Delta = 1$ that we constructed is $p(6)+p(5)-p(2)-p(0)$. One can then verify using the explicit values of $p(i)$ that this number is equal to $p(7)$ the total expected number of hwv's at that level. Thus again all the physical states are null.

The counting in the construction of null states is now obvious. All one has to take into account are the representations that have null states which are separated into parallel non communicating pairs, the type I with dimensions $\Delta_I^1(k) = -k(6k-5)$, $\Delta_I^2(k) = (1-k)(6k-1)$ and the type II with $\Delta_{II}^1(k) = (k+1)(4-6k)$, $\Delta_{II}^2(k) = (1-k)(6k+4)$ where $k \in Z^+$. One starts with the first one or two representations that have null vectors and starts using them to separate the states in the next two such representations into null states and states with positive norm. Each of the states with positive norm will give one null hwv in the next two representations down the line, whereas the already null hwv's will contribute null states equal to half their number and so on. Thus the number R^N_{null} of linearly independent null hwv's at level N in the expansion (6) is given by

$$R^N_{null} = \sum_{\substack{n=\Delta^{1,2}_{II}(k) \\ k\in Z^+}}^{0\le n\le N-1} p(n) - \sum_{\substack{n=\Delta^{1,2}_{I}(k) \\ k\in Z^+}}^{0\le n\le N-1} p(n) \tag{8}$$

Then Euler's pentagonal identity,[§]

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{n\in Z}(-1)^n q^{\frac{3n^2+n}{2}} \tag{9}$$

implies that $R^N_{null} = p(N)$! Thus all the physical states at level N are all null.

What we showed so far is that if we couple a conformal field theory to two-dimensional gravity then the physical states are in one to one correspondence with the Virasoro primary fields of the conformal field theory[¶]. This result is also true for (1,1) and (1,0) supergravity coupled to matter, and the same reasoning goes through. In the context of the discreet approach to gravity the physical states of two-dimensional gravity coupled to minimal models fall into two classes. The ones which correspond to diagonal entries in the Kač formula are singled out, whereas the rest behave as gravitational descendants. Such behavior is not obvious from the continuum approach to gravity and its representation theory. It would be interesting to understand their behaviour from general principles of the Virasoro algebra.

[§]Euler's pentagonal identity is equivalent to the statement that the character of the irreducible Virasoro representation with $c = 0$, $\Delta = 0$ is equal to 1, [9].

[¶]We should mention here that our proof holds in general, irrespective of unitarity or reducibility of the the initial representations.

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