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Structure and Representations of the W_∞ Algebra

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Abstract

We present an up to date review of the structure and representations of the W_∞ algebra. The central charge is constrained by unitarity to take the values $c = 2p$, with $p = 1, 2, 3, \cdots$. We also extend our discussion to W_∞^p, which is a $U(p)$-matrix generalization of W_∞, using the symmetries of the Grassmannian coset models $SU(p+1)_N/SU(p)_N \otimes U(1)$ in the large N limit.

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1. Introduction

The main subject of our paper is the chiral operator algebra W_∞, which is a quantum deformation of the infinite dimensional Lie algebra,

$$[W_m^s, W_n^{s'}] = ((s'-1)m - (s-1)n)W_{m+n}^{s+s'-2} \tag{1.1}$$

with linear terms in the W-generators as well as central terms. All indices assume integer values with the restriction that $s, s' \geq 2$. We examine the occurrence of this symmetry in 2-d Conformal Field Theory, (CFT) and construct unitary representations of W_∞ using free bosons. The present work is a review of the main ideas and results we have obtained recently in references [1, 2, 3].

The classical algebra (1.1) is interesting from the point of view of CFT, because it contains the (centerless) Virasoro algebra

$$[W_m^2, W_n^2] = (m-n)W_{m+n}^2 \tag{1.2}$$

as a subalgebra. In this respect, (1.1) may be viewed as an extended conformal algebra with an infinite collection of additional symmetries generated by $\{W_n^s;\ n \in Z\}$ with $s \geq 3$. The conformal properties of these additional generators follow from the commutation relations

$$[W_m^2, W_n^s] = ((s-1)m - n)W_{m+n}^s \ , \tag{1.3}$$

which imply that W_n^s can be regarded as the Fourier modes of primary fields with conformal weight s. Therefore, the algebra (1.1) describes (at the classical level) higher spin transformations in two dimensions with $s = 2, 3, 4, \cdots$.

On the other hand, the algebra (1.1) has a natural geometric interpretation as area preserving diffeomorphisms of 2-manifolds. It can be represented by the Poisson bracket of functions $W_n^s = x^{n+s-1}y^{s-1}$ on a two dimensional plane with (canonical) coordinates x and y,

$$\{x, y\} = 1 \ , \tag{1.4}$$

or equivalently by the smooth functions $e^{inx}y^{s-1}$ on the cylinder $R \times S^1$. This symmetry assigns a definite meaning to higher spin transformations of CFT. However it should be emphasized that the area preserving diffeomorphism algebra does not refer to the 2-d world of CFT, since otherwise it would be incompatible with chiral conformal transformations. For this reason it is more appropriate to introduce an additional space-time dimension in order to interpret the extended conformal algebra (1.1) geometrically. Then, the Virasoro subalgebra (1.2) is generated by point canonical transformations in the (x, y)-space.

The complete structure of W_∞, which is defined as a large N limit of Zamolodchikov's W_N operator algebra, [4], is a deformation of the area preserving diffeomorphism symmetry (1.1). In particular, for any given s and s', the commutation relations of the area preserving diffeomorphism algebra and W_∞ differ from each other by local functionals of the generating fields with spin less that $s+s'-2$. Since both infinite dimensional algebras satisfy the Jacobi identity (associativity), the deformation terms cannot be arbitrary; they are 2-cocycles of the Lie algebra (1.1) with non-trivial coefficients in general. The existence of consistent gauge interactions among higher spin fields with all integer values of spin, $s \geq 2$, suggests that the deformation terms should be central or linear, but not quadratic (or higher polynomial) in the W-fields. A deformation of this type was introduced from purely algebraic considerations by Pope, Romans and Shen (PRS), [5] and its uniqueness was investigated in [6]. Subsequent work on the large N limit of W_N-minimal models proved that the PRS commutation relations provide the correct description of the W_∞ algebra. It is our purpose to review these results in detail and present an account of the structure and field-theoretic representations of W_N algebras in the large N limit.

In section 2 we use the Hamiltonian formalism of Gelfand and Dickey to describe the universal features of W-algebras at large N and their relation with the area preserving diffeomorphism symmetry (1.1). In section 3 we introduce the PRS deformation of the commutation relations (1.1) and discuss its physical meaning in the theory of Z_∞ parafermions. In section 4 we use a collection of p complex free bosons in 2-d to construct explicit field-theoretic realizations of the W_∞ algebra with central charge $c=2p$. In section 5 we show that the representation theory of $U(1)^{2p}$ current algebras yields highest weight (hw) unitary irreducible representations of W_∞ with $c=2p$. In section 6 we obtain a $U(p)$ matrix generalization of W_∞ and investigate its structure in the limit $p\to\infty$. It is found that the resulting symmetry describes symplectic diffeomorphisms in four dimensions. Finally, section 7 contains our conclusions and further comments.

2. The universal features of W-algebras

In CFT, W_N-algebras are generated by the stress tensor $T(z)=W^2(z)$ and a collection of additional conserved (chiral) fields $\{W^s(z)\,;\, s=3,4,\cdots,N\}$ with spin s, [4]. They form closed operator algebras with non-linear determining relations in general. Introducing Fourier modes, $W^s(z)=\sum_{n\in Z} W^s_n z^{-n-s}$, their commutation relations assume the form

$$[W^2_m, W^2_n] = (m-n)W^2_{m+n} + \frac{c}{12}(m^3-m)\delta_{m+n,0} \ , \tag{2.1}$$

$$[W^2_m, W^s_n] = ((s-1)m-n)W^s_{m+n} \ , \tag{2.2}$$

$$[W^s_m, W^{s'}_n] = \sum_{\{s_i\},\{k_i\}} C^{ss'}_{s_1 s_2\cdots s_p}(m,n;k_1,k_2\cdots,k_p;c)W^{s_1}_{k_1}\cdots W^{s_p}_{k_p} \ , \tag{2.3}$$

where $k_1+k_2+\cdots+k_p=m+n$, $W^0_k=\delta_{k,0}$ (inclusion of the identity operator that accounts for central terms) and $s_1+s_2+\cdots+s_p\leq s+s'-2$. For higher spin fields the structure constants $C^{ss'}_{\{s_i\}}$ are not universal in the sense that many of them depend implicitly on N and this makes their computation highly non-trivial. However, taking a suitable limit in which $N \to \infty$, the structure of W-algebras simplifies considerably and the commutation relations of the resulting infinite dimensional symmetry algebra (when appropriately defined) are determined only by universal constants. The task is to extract the universal features of higher spin transformations in 2-d.

Our starting point is the Gelfand-Dickey algebra of formal pseudo-differential operators which provides a (classical) hamiltonian framework for W_N symmetries, [7, 8]. Using this formalism we will describe the connection of W_∞ with the area preserving diffeomorphism algebra (1.1). Let

$$L_N = \partial_z^N + u_2(z)\partial_z^{N-2} + \cdots + u_{_N}(z) \tag{2.4}$$

be an N-th order differential (Lax) operator with N-1 coordinate (potential) functions $\{u_i(z)\ ; i = 2, 3, \cdots, N\}$. We also consider (formal) pseudo-differential operators $A(z) = A_-(z) + A_+(z)$ with

$$A_-(z) = \sum_{k \in Z^-} \partial_z^k \alpha_k(z)\ \ ;\ \ A_+(z) = \sum_{k \in Z_0^+} \alpha_k(z)\partial_z^k \tag{2.5}$$

and introduce the notation $resA = \alpha_{-1}(z)$. Then, to any local functional $f[u_i]$, we assign the formal operator sum

$$X_f = \sum_{k=1}^{N-1} \partial_z^{-k} \frac{\delta f}{\delta u_{_{N+1-k}}} + \partial_z^{-N} x_{_N}(f)\ \ . \tag{2.6}$$

The variable $x_{_N}(f)$ is chosen so that the condition $res[L_{_N}, X_f] = 0$ is satisfied. With this in mind, the Gelfand-Dickey bracket between any two functionals $f[u]$ and $g[u]$ is defined to be

$$\{f, g\}_{_N} = \int res[V_{X_f}(L_{_N})X_g]\ \ , \tag{2.7}$$

where

$$V_{X_f} = L_{_N}(X_f L_{_N})_+ - (L_{_N} X_f)_+ L_{_N}\ \ . \tag{2.8}$$

In all formulae we use Leibnitz's rule for the multiplication of operators (both differential and formal).

Under the Gelfand-Dickey bracket (2.7), the coordinate functionals $u_2, \cdots, u_{_N}$ form a closed algebra with quadratic determining relations. The Virasoro subalgebra is generated by $u_2(z)$

$$\{u_2(z), u_2(z')\}_{_N} = (u_2(z) + u_2(z'))\delta'(z - z') + \frac{c}{12}\delta'''(z - z') \tag{2.9}$$

and has central charge $c = N^3 - N$. However, the rest of the coordinate fields $u_s(z)$ with $s = 3, 4, \cdots, N$, are not primary. Primary conformal fields $W^s(z)$ with spin s are obtained using suitable polynomial combinations of all the u_i and their derivatives. They are of the general form

$$W^s(z) = \sum_{\{i\},\{k\}} A^{i_1\cdots i_p}_{N;k_1\cdots k_p} u^{(k_1)}_{i_1}(z)\cdots u^{(k_p)}_{i_p}(z) \tag{2.10}$$

with $k_1+\cdots+k_p+i_1+\cdots+i_p = s \leq N$. The fields $W^s(z)$, including $W^2(z)=u_2(z)$, generate at the classical level the extended conformal algebra W_N. Its quantum analog is obtained by introducing appropriate normal orderings which regularize the composite (polynomial) terms and deform the structure constants of the algebra accordingly, while maintaining associativity.

In this classical framework, the commutation relations of W_N involve structure constants which diverge rapidly in the limit $N\to\infty$. We may avoid this behavior by modifying the definitions of the bracket (2.7) and the u-variables in a suitable (and consistent) way. For this reason we introduce the rescaling*

$$[f,g]_{_N} \equiv N^3\{f,g\}_{_N} \ \ , \tag{2.11}$$

$$U_s(z) \equiv N^{-\frac{3s}{2}} u_s(z) \tag{2.12}$$

and then take the large N limit of the Gelfand-Dickey algebra. With this prescription in mind, we find that the commutation relations $[U_s(z), U_{s'}(z')]_\infty$ are well-defined and involve no infinities for all $s, s' \geq 2$. Explicit expressions are available in ref. [2] and involve linear and quadratic terms in the U-variables. However, it is possible to introduce a tower of higher spin fields

$$W^2(z) = U_2(z) \ \ , \ \ W^3(z) = U_3(z) \ \ , \tag{2.13a}$$

$$W^4(z) = U_4(z) - \frac{1}{2}U_2^2(z) \ \ , \tag{2.13b}$$

$$W^5 = U_5(z) - U_2(z)U_3(z) \ \ , \tag{2.13c}$$

$$W^6(z) = U_6(z) - U_2(z)U_4(z) - \frac{1}{2}U_3^2(z) + \frac{1}{3}U_2^3(z) \ \ , \tag{2.13d}$$

etc., so that the commutation relations $[W^s(z), W^{s'}(z')]_\infty$ become linear and $W^s(z)$ with $s\geq 3$ become primary.

More explicitly we define fields $W^s(z)$ using the generating function

$$W(t) = \log(1+U(t)) \ \ , \tag{2.14}$$

where

$$W(t) = \sum_{s=2}^{\infty} W^s(z)t^s \ \ , \ \ U(t) = \sum_{s=2}^{\infty} U_s(z)t^s \tag{2.15}$$

*This rescaling is chosen so that in the classical version of the Feigin-Fuks realization of W_N-algebras (known as the Miura transformation) the commutation relations of the free-$U(1)$ fields remain unchanged.

and $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$. Then, the Gelfand-Dickey commutation relations of the W-fields are given by

$$[W^s(z), W^{s'}(z')]_\infty = \Big((s-1)W^{s+s'-2}(z) + (s'-1)W^{s+s'-2}(z')\Big)\,\delta'(z-z') \ , \qquad (2.16)$$

up to central terms in the Virasoro subalgebra only, [2]. Introducing Fourier modes W^s_n $(n \in Z)$, we obtain precisely the commutation relations of the area preserving diffeomorphism algebra (1.1). The results above can be derived by straightforward but lengthy computations.

The most systematic way to construct the quantum version of W_N algebras is given by the (Feigin-Fuks) free-field representation of the W-generators. The idea is to factorize the Lax operator (2.4) as $\prod_{i=1}^N(\partial_z + q_i(z))$, treat the q-variables as $U(1)$ currents and then compute the commutation relations of the u-fields by introducing suitable normal orderings, [4]. To be more precise, let $\vec{\phi} = (\phi_1, \phi_2, \cdots, \phi_{N-1})$ be an $(N-1)$-vector of free massless scalar fields with two-point functions

$$\langle \phi_i(z)\phi_j(w)\rangle = -2\delta^{ij}\log(z-w) \qquad (2.17)$$

and consider the normal-ordered differential (Lax) operator

$$L_N =: \prod_{m=1}^N (i\alpha_0\partial_z + \vec{h}_m \cdot \partial_z\vec{\phi}(z)) : \ . \qquad (2.18)$$

The numerical coefficient α_0 is the background charge of the theory and the N vectors $\vec{h}_m$ are chosen so that

$$\sum_{m=1}^N \vec{h}_m = 0 \ , \ 2\vec{h}_m \cdot \vec{h}_n = \delta^{mn} - \frac{1}{N} \ . \qquad (2.19)$$

Using the Leibnitz rule, if we rewrite the product (2.18) in the form $L_N = \sum_{s=0}^N u_s(z)(i\alpha_0\partial_z)^{N-s}$, we will obtain a free field realization for the u-fields. Then, the two-point functions (2.17) determine unambiguously the (quantum) commutation relations of the chiral operator algebra W_N.

The expressions we obtain this way are similar to the classical Gelfand-Dickey relations. However, the quantization of the fields $u_s(z)$ leads to a deformation of the structure constants, which also depend in this case on the background charge α_0. Introducing Fourier modes $u_s(m)$, Fateev and Lykyanov obtained the following explicit results, [4],

$$[u_2(m), u_s(n)] = (m(s-1)-n)u_s(m+n) - \sum_{k=1}^s (i\alpha_0)^k \frac{(N-s+k)!}{(N-s)!} A^s_{m,k} u_{s-k}(m+n) \ , \quad (2.20)$$

$$[u_3(0), u_s(m)] = 2m(\delta^{s,N} - 1)u_{s+1}(m) + i\alpha_0(2s+m-3)mu_s(m) -$$

$$-\frac{2}{N}\sum_{k=1}^{s}(i\alpha_0)^{k-1}\frac{(N-s+k)!}{(N-s)!}\left[\sum_{n\in Z} C_k^{n-m} : u_{s-k}(n)u_2(m-n) : +B^s_{m,k}u_{s-k}(m)\right] . \quad (2.21)$$

Here, the central charge of the Virasoro subalgebra is

$$c = (N-1)[1 - N(N+1)\alpha_0^2] \quad (2.22)$$

and the coefficients A, B, C are given by

$$A^s_{m,k} = C^{m+1}_{k+1}\left(\frac{k-1}{2N\alpha_0^2} - s + 1 - \frac{(N-1)(k-1)}{2}\right) , \; C^m_k = \binom{m}{k} \quad (2.23)$$

$$B^s_{m,k} = (-1)^k m C^{k+1-i}_{k+1} + \left((-1)^k C^{k-1+i}_{k+2} - C^{i+1}_{k+2}\right)\left(\frac{k-1}{2N\alpha_0^2} - s + 1 - \frac{(N-1)(k-1)}{2}\right), \quad (2.24)$$

where $i = [\frac{m+1}{2}]$. The rest of the commutation relations of the (quantum) W_N algebra are much more complicated to derive explicitly and there are no closed formulae for arbitrary higher spin in our immediate disposal.

If we keep the background charge α_0 fixed (and independent of N), the quantum corrections to the structure constants of the W_N algebra are negligible compared to the (classical) Gelfand-Dickey coefficients in the limit $N \to \infty$. This can be easily seen by inspection of the commutation relations (2.20), (2.21) and the value of the central charge (2.22). Bilal proved in general that when $\alpha_0^2 \sim 1$, all quantum commutation relations of the W_N algebra reduce to their classical counterparts as $N \to \infty$, [9]. However, without any rescaling, the structure constants of the algebra are divergent in this limit. On the other hand, Morozov considered the "pure quantum" limit of W_N algebras in which $N \to \infty$, but $\alpha_0 = 0$, [10]. In his case, the central charge of the Virasoro subalgebra diverges linearly for large values of N.

¿From these considerations it is quite clear that the quantum commutation relations of W_∞ are not uniquely defined and the structure of the universal W-algebra which arises in the large N limit depends on how α_0 scales with N. However, if the background charge α_0 is chosen to be $\sim 1/N$, all structure constants in eqs. (2.20), (2.21) will be finite at large N and no rescaling is necessary. In this limit, and in analogy with the classical Gelfand-Dickey calculation, we may introduce new variables $W^s(z)$ in order to eliminate the quadratic terms from the commutation relations of W_∞. In fact the field redefinitions we used earlier are also applicable in the present case, provided that normal orderings are taken into account. We define new fields $W^s(z)$ using (without any rescaling) the generating function

$$W(t) =: \log(1 + u(t)) : \; , \quad (2.25)$$

which in terms of Fourier modes gives

$$W^2_m = u_2(m) \; , \; W^3_m = u_3(m) \; , \quad (2.26a)$$

$$W_m^4 = u_4(m) - \frac{1}{2} \sum_{n \in Z} : u_2(n) u_2(m-n) : \ , \qquad (2.26b)$$

etc., in exact analogy with eqs. (2.13). Then, their commutation relations will be given by eq. (2.16) (or equivalently eq. (1.1)), up to subleading terms which involve only derivatives of the fields $W^s(z)$ as well as derivatives of δ-functions. The important point here is that under the redefinitions (2.25), all purely polynomial terms will be absent from the quantum commutation relations of W_∞, irrespectively of the choice of α_0. This is so, because the structure constants of the purely polynomial u-terms in the commutation relations of W_N algebras are independent of the background charge and do not deform upon quantization.† Since the rescaling (2.11) and (2.12) does not change the structure constants of purely polynomial terms, the results of our previous analysis are valid for them, in all cases.

We emphasize that different limiting procedures for the quantum commutation relations of W_N at large N (depending on the choice of α_0) do not affect the leading structure (1.1), but only the subleading terms associated with derivatives of the fields $W^s(z)$ and central terms. These subleading terms give rise to a deformation of the area preserving diffeomorphism algebra (1.1). However, it is quite difficult to compute them directly in closed form, using the Feigin-Fuks representation. Moreover, even after the redefinitions (2.25), there is no guarantee that the deformation terms will assume a simple linear form; they might involve contributions from derivative terms like $(\partial W^2(z))^2$, etc. It is possible to show by sample calculations that in the limit $N \to \infty$ (with $\alpha_0 \sim 1/N$), such non-linear subleading terms can be eliminated as well in a consistent way. For this we have to introduce further field redefinitions by adding certain derivative terms in the expressions (2.26) for our basic fields $W^s(z)$. Since there are no closed formulae for the quantum commutation relations of W_N with arbitrary N (apart from (2.20) and (2.21)), it is practically impossible to carry out this procedure systematically for all higher spin fields.

In the next section we adopt an ansätz for the complete structure of W_∞, introduced by Pope, Romans and Shen from purely algebraic considerations by requiring linearity and compatibility with the Jacobi identity, [5]. Since there is no a priori way to single out one algebraic deformation of (1.1) from the others, we appeal to the representation theory of the symmetry algebra in question for a definite physical realization of the commutation relations of W_∞. In particular, we know that the commutation relations of W_∞ should be realized by the CFT of Z_∞ parafermions (which is well-defined). In this framework, we prove that the large N limit of the quantum W_N algebras, such that the central terms remain finite, is given precisely by the (linear) PRS algebra. This is one of our main motivations for being interested in the field theoretic realizations of W_∞ which

†One may verify this by using the (Feigin-Fuks) free field representation, $u_s(z) \sim (\partial\phi)^s$+derivative terms, after performing simple contractions only.

we discuss in detail in the next sections.

3. The PRS deformation and Z_∞ parafermions

The PRS algebra is a linear deformation of the area preserving diffeomorphism symmetry, which also admits central terms in the commutation relations of higher spin fields. It has the form

$$[W_m^s, W_n^{s'}] = ((s'-1)m - (s-1)n)W_{m+n}^{s+s'-2} + q^{2(s-2)}c_s(m)\delta^{s,s'}\delta_{m+n,0}+$$
$$+q^2 g_2^{ss'}(m,n)W_{m+n}^{s+s'-4} + q^4 g_4^{ss'}(m,n)W_{m+n}^{s+s'-6} + \cdots \quad , \tag{3.1}$$

where the coefficients of the central terms are

$$c_s(m) = \frac{c}{2}m(m^2-1)(m^2-4)\cdots(m^2-(s-1)^2)\frac{2^{2(s-3)}s!(s-2)!}{(2s-1)!!(2s-3)!!} \tag{3.2}$$

and the sequence of $\cdots$ terms in (3.1) terminates with W_{m+n}^2 for $s+s'$ even and with W_{m+n}^3 for $s+s'$ odd. The coefficients $g^{ss'}(m,n)$ are polynomial expressions in m, n given by the formulae

$$g_l^{ss'}(m,n) = \frac{\varphi_l^{ss'}}{2(l+1)!}N_l^{ss'}(m,n) \quad , \tag{3.3}$$

where

$$\varphi_l^{ss'} = \sum_{k\in Z_0^+} \frac{(-\frac{1}{2})_k(\frac{3}{2})_k(-\frac{l+1}{2})_k(-\frac{l}{2})_k}{k!(-s+\frac{3}{2})_k(-s'+\frac{3}{2})_k(s+s'-l-\frac{3}{2})_k} \quad , \tag{3.4}$$

$$N_l^{ss'}(m,n) = \sum_{k=0}^{l+1}(-1)^k\binom{l+1}{k}(2s-l-2)_k[2s'-k-2]_{l+1-k}\cdot$$
$$\cdot[s-1+m]_{l+1-k}[s'-1+n]_k \tag{3.5}$$

and

$$(a)_k \equiv a(a+1)(a+2)\cdots(a+k-1) \quad , \tag{3.6a}$$
$$[a]_k \equiv a(a-1)(a-2)\cdots(a-k+1) \quad , \tag{3.6b}$$

$(a)_0 = [a]_0 = 1$.

The parameter c is the central charge of the Virasoro algebra and its value depends on the underlying CFT. The second deformation parameter q is arbitrary. For $q=0$, all subleading terms in the commutation relations of the PRS algebra vanish with the exception of central terms in the Virasoro subalgebra. It can be verified directly that this is the only possible central extension of the area preserving diffeomorphism algebra (1.1) which is consistent with the Jacobi identities. Central terms for higher spin fields are allowed provided that $q \neq 0$. In this case we may rescale the generators W_n^s by q^{s-2} and normalize the value of the q-parameter to one. ¿From now on we choose to work with $q=1$ without loss of generality. For completeness we point out that for $c=0$ but

$q \neq 0$, the algebra (3.1) is essentially the same (up to variable redefinitions) with the algebra of (symbols of) differential operators on the line, also known as Moyal algebra. It has been established in the context of elementary quantum mechanics that the Moyal algebra is the unique (linear) deformation of the area preserving diffeomorphism algebra, (see for instance [11] and references therein). The importance of the Moyal (sine bracket) algebra in large N limit considerations, was first recognized by Fairlie and Zachos. Their results, which are applicable to $SU(N)$, will be discussed in section 6 in connection with the algebra W_∞^∞. However, the possibility to have non-zero central terms for all higher spin generators in the commutation relations (3.1), is due to Pope, Romans and Shen.

The PRS algebra was originally introduced in the literature as a candidate for W_∞, from purely algebraic considerations alone. However, a closer look at the minimal models of W_N algebras shows that in the large N limit, the commutation relations of W_N become linear and the complete structure of W_∞ is given exactly by (3.1). This provides the physical justification for identifying the PRS algebra with W_∞. As a byproduct we obtain unitary hw irreducible representations of W_∞, which might also be of some value in mathematics. These results were first presented in ref. [3] and will be the subject of the remaining of this paper.

Recall that for W_N algebras there is a series of unitary CFT models which are minimal in the sense that the corresponding number of W_N representations is finite, [4]. These theories are parametrized by a positive integer p and possess a (global) $Z_N \otimes Z_N$ symmetry. The central charges of W_N-minimal models are given by the sequence[‡]

$$c_p^N = (N-1)\left[1 - \frac{N(N+1)}{(N+p)(N+p+1)}\right] \tag{3.7}$$

and can be identified with the coset models

$$\frac{SU(N)_1 \otimes SU(N)_p}{SU(N)_{p+1}} \quad . \tag{3.8}$$

The simplest example is provided by the theory of Z_N parafermions which corresponds to $p = 1$ and has central charge $c = 2(N-1)/(N+2)$, [12]. We will discuss some aspects of this model in detail in order to establish the linearization of the W_N commutation relations in the large N limit.

For convenience, we adopt an alternative description of the coset models (3.8), which avoids unnecessary complications dealing with the large N limit of $SU(N)$. Instead we will study the Grassmannian coset models

$$G_N(p) = \frac{SU(p+1)_N}{SU(p)_N \otimes U(1)} \quad . \tag{3.9}$$

[‡]Notice that this is consistent with the choice of the background charge $\alpha_0 \sim 1/N$ we adopted in section 2 (cf. eq. (2.22)).

The models in (3.8) and (3.9) have the same central charge, the same chiral algebra, W_N and the same hw irreducible representations and therefore provide two equivalent descriptions of the minimal models of W_N algebras for all $N \geq 2$. For $p = 1$ we have the usual parafermionic coset, $G_{_N}(1) = SU(2)_{_N}/U(1)$. The Z_N parafermions are defined through their relation with the level-N $SU(2)$ current algebra, [13]

$$J^a(z)J^b(w) = \frac{N}{2}\frac{\delta^{ab}}{(z-w)^2} + i\varepsilon^{abc}\frac{J^c(w)}{(z-w)} \quad . \tag{3.10}$$

In particular, we set

$$J^3 = i\sqrt{N/2}\,\partial\rho \ , \ J^{\pm} = \sqrt{N} exp\left(\pm\sqrt{\frac{2}{N}}i\rho\right)\psi_{\pm 1} \ , \ \langle\rho(z)\rho(w)\rangle = -\log(z-w) \tag{3.11}$$

and the operator product expansion (OPE) of the parafermions assumes the form

$$\psi_k(z)\psi_{k'}(w) = c_{k,k'}(z-w)^{-\frac{2kk'}{N}}\left[\psi_{k+k'}(w) + \mathcal{O}(z-w)\right] \ , \tag{3.12}$$

$$\psi_k(z)\psi_{-k}(w) = (z-w)^{-\frac{2k(N-k)}{N}}\left[1 + \frac{k(N-k)(N+2)}{N(N-1)}(z-w)^2 T_\psi(w) + \cdots\right] \ , \tag{3.13}$$

where

$$\psi_k^\dagger = \psi_{-k} \ , \ \psi_{N+k} = \psi_k \ , \ \psi_0 = 1 \ , \tag{3.14a}$$

$$c_{k,k'} = \left[\frac{(k+k')!(N-k)!(N-k')!}{k!k'!N!(N-k-k')!}\right]^{\frac{1}{2}} \tag{3.14b}$$

and T_ψ is the parafermionic stress tensor.

The spin of the parafermions ψ_k is given by $\Delta_k = \frac{k(N-k)}{N}$ for $0 \leq k \leq N-1$ and it is generically fractional. For this reason the parafermionic algebra is non-local in general. However, for $N = 2$, the spin is 1/2 and the parafermion ψ_1 is nothing else but an ordinary Mayorana-Weyl fermion. The only other case where the parafermionic algebra becomes local is in the limit $N \to \infty$. Then, the scaling dimensions of $\psi_{\pm k}$ take all positive integer values, $\Delta_k = k(1 - \frac{k}{N}) \to k$ and the discrete $Z_N \otimes Z_N$ symmetry is promoted to affine $U(1) \otimes U(1)$. Therefore, it is natural to expect that W_N simplifies considerably at large N and admits a realization in terms of free bosons. It is a trivial observation that as $N \to \infty$, the $SU(2)$ current algebra "flattens" becoming an abelian $U(1)^3$ current algebra. This follows by a simple rescaling of the $SU(2)$ currents in (3.10) as $J^a(z) \to N^{-\frac{1}{2}}J^a(z)$. This way, the large level limit of the current algebra is well defined and involves no infinities. Considering the coset $SU(2)_\infty/U(1)$ means removing the scalar field ρ from the spectrum and identifying the other generators $\psi_{\pm 1} = \lim_{N\to\infty} N^{-\frac{1}{2}}J^{\pm}(z)$ (cf. eq. (3.11)) with the remaining $U(1)$ currents, $i\partial_z\phi$ and $i\partial_z\bar{\phi}$ respectively. Then, the parafermions with $k \geq 2$ are expressed as composite operators in the $U(1)$ currents, $\psi_k \sim (\partial\phi)^k$ and $\psi_k^\dagger \sim (\partial\bar{\phi})^k$.

It is quite clear now that the algebra (3.12), (3.13) for Z_∞ parafermions is identical with the enveloping algebra of the $U(1)\otimes U(1)$ current algebra. Moreover, the generators of the W_N chiral algebra admit a simple realization in the large N limit in terms of the $U(1)$ currents $\partial\phi$ and $\partial\bar{\phi}$. It is well known that for all N, the stress tensor and the other higher spin fields of W_N appear in the non-singular terms of the OPE $\psi_1(z)\psi_1^\dagger(w)$, (cf. eq. (3.13)). In the limit $N\to\infty$, the latter OPE is equivalent to $\partial_z\phi(z)\partial_w\bar{\phi}(w)$ and therefore only terms bilinear in the scalar fields will be generated. Thus, the subleading (non-singular) terms that appear in this case will be of the form $\partial^k\phi\partial^l\bar{\phi}$. The operator $\partial\phi\partial\bar{\phi}$ is proportional to the stress tensor of the theory, which has central charge $c_1^\infty = 2$. The remaining generators $W^s(z)$ can be identified with linear combinations of the operators $\partial^k\phi\partial^l\bar{\phi}$ with $k+l=s$. Clearly, their operator algebra, which generates W_∞, is closed and involves linear terms only.

The analysis above proves the linearization of the W_N commutation relations in the large N limit, without any explicit calculations. In order to compute the structure constants of the W_∞ algebra, we have to perform the OPE for the fields $W^s(z)$ using their bilinear expressions in terms of the $U(1)$ currents, $\partial\phi$ and $\partial\bar{\phi}$. Notice that for any given spin $s\geq 2$ there are $s-1$ different operators of the form $\partial^k\phi\partial^l\bar{\phi}$ with the appropriate scaling dimension. Since all of them but one can be expressed as derivatives of lower spin fields , it is sufficient to choose one representative for every value of s. Different choices are linearly related to each other and do not change the structure of the W_∞ algebra. However, there is a unique choice of basis in which the generators $W^s(z)$ are quasiprimary. In the next section we construct this basis and prove that the commutation relations of W_∞ are given by the PRS algebra (3.1).

4. Bosonic realizations of W_∞

Let us consider a free complex scalar field in 2-d with two-point function

$$\langle\phi(z)\phi(w)\rangle = \langle\bar{\phi}(z)\bar{\phi}(w)\rangle = 0 \ , \ \langle\phi(z)\bar{\phi}(w)\rangle = -\log(z-w) \ . \tag{4.1}$$

We also introduce the following bilinear expressions

$$W^s(z) = B(s)\sum_{k=1}^{s-1}(-1)^k A_k^s \partial_z^k\phi\partial_z^{s-k}\bar{\phi} \tag{4.2}$$

for all $s\geq 2$, where the numerical coefficients $B(s)$ and A_k^s are arbitrary for the moment. Normal orderings are implicitly assumed throughout this section. The requirement that the fields (4.2) are quasiprimary fixes uniquely the relative coefficients A_k^s, up to an overall normalization constant which we denote by $B(s)$. Since we assume that the W_∞ symmetry is unbroken, ie. $\langle W^s(z)\rangle = 0$ for all s, the condition that the fields (4.2) are quasiprimary takes the form

$$\langle W^s(z)W^{s'}(w)\rangle \sim \frac{\delta^{s,s'}}{(z-w)^{s+s'}} \ . \tag{4.3}$$

Explicit calculation shows that in our case, eq. (4.3) becomes

$$\langle W^s(z)W^{s'}(w)\rangle = B(s)B(s')\frac{I(s,s')}{(z-w)^{s+s'}} \quad , \tag{4.4}$$

where

$$I(s,s') = (-1)^s \sum_{k=1}^{s-1}\sum_{l=1}^{s'-1}(-1)^{k+l}(s+l-k-1)!(s'+k-l-1)!A_k^s A_l^{s'} \quad . \tag{4.5}$$

The condition $I(s,s') = 0$ for $s \neq s'$ puts severe constraints on A_k^s. We find that there is a unique solution to these conditions (up to normalization) given by

$$A_k^s = \frac{1}{s-1}\binom{s-1}{k}\binom{s-1}{s-k} \quad . \tag{4.6}$$

To prove this result it is sufficient to show that $I(s,s') = 0$ for $s' = 2, 3, \cdots, s-1$ and for all $s > 2$. We also point out that the coefficients A_k^s appear in the definition of the (1,1) Jacobi polynomials, $P_{s-2}^{(1,1)}(x)$ and the value of $I(s,s')$ in (4.5) follows directly from the orthogonality relations of these polynomials.

The solution (4.6) enjoys the property $A_k^s = A_{s-k}^s$, which implies that the operators $W^s(z)$ in (4.2) are even (odd) under the interchange $\phi \leftrightarrow \bar{\phi}$ for s even (odd). In view of this symmetry the OPE of W^s with $W^{s'}$ will involve $W^{s''}$ with $s'' = s+s'-2$, $s+s'-4$, $s+s'-6, \cdots$ only, which is the main feature of the PRS algebra. Up to an overall constant $B(s)$ that will be determined later, the quasiprimary fields assume the form

$$W^2(z) = -\partial_z\phi\partial_z\bar{\phi} \quad , \tag{4.7a}$$

$$W^3(z) = -(\partial_z\phi\partial_z^2\bar{\phi} - \partial_z^2\phi\partial_z\bar{\phi}) \quad , \tag{4.7b}$$

$$W^4(z) = -(\partial_z\phi\partial_z^3\bar{\phi} - 3\partial_z^2\phi\partial_z^2\bar{\phi} + \partial_z^3\phi\partial_z\bar{\phi}) \quad , \tag{4.7c}$$

$$W^5(z) = -(\partial_z\phi\partial_z^4\bar{\phi} - 6\partial_z^2\phi\partial_z^3\bar{\phi} + 6\partial_z^3\phi\partial_z^2\bar{\phi} - \partial_z^4\phi\partial_z\bar{\phi}) \quad , \tag{4.7d}$$

etc. In this tower of higher spin fields only $W^3(z)$ is primary.

The commutation relations of W_∞ can be determined in this basis by computing the OPE $W^s(z)W^{s'}(w)$, for all $s, s' \geq 2$. We find that

$$W^s(z)W^{s'}(w) = B(s)B(s')\left[\frac{(2s-2)!}{s(s-1)}\frac{\delta^{s,s'}}{(z-w)^{s+s'}} + \sum_{l=1}^{s+s'-2}\frac{R_l^{ss'}(\partial_w\phi,\partial_w\bar{\phi})}{(z-w)^l}\right], \tag{4.8}$$

where

$$R_l^{ss'}(\partial\phi,\partial\bar{\phi}) = \sum_{k=1}^{s-1}\sum_{k'=1}^{s'-1}(-1)^{k'}A_k^s A_{k'}^{s'}\frac{(k+k'-1)!}{(k+k'-l)!}[(-1)^s\partial^{s+k'-l}\phi\partial^{s'-k'}\bar{\phi}+$$

$$+(-1)^{s'}\partial^{s'-k'}\phi\partial^{s+k'-l}\bar{\phi}]. \tag{4.9}$$

Then, we may express the operators (4.9) as linear combinations of the W-fields and their derivatives and use the standard formula

$$[W^s_m, W^{s'}_n] = \oint_{C_0} \frac{dw}{2\pi i} w^{n+s'-1} \oint_{C_w} \frac{dz}{2\pi i} z^{m+s-1} W^s(z) W^{s'}(w) \tag{4.10}$$

to obtain the commutation relations of W_∞ in terms of Fourier modes. The explicit calculations are rather involved for arbitrary s and s'. Making heavy use of symbolic manipulation we have verified that the algebra (4.10) is identical to (3.1), provided that the normalization constant $B(s)$ is chosen to be

$$B(s) = \frac{2^{s-3} s!}{(2s-3)!!} \quad . \tag{4.11}$$

This completes the argument that the W_∞ algebra is described by the PRS commutation relations.

In our case the value of the central charge is $c = 2$ because we have considered the simplest unitary model of the W_∞ algebra. However, it is straightforward to generalize the bosonic realization (4.2) to W_∞ with $c = 2p$, where p is a positive integer. For this we introduce p independent complex free scalar fields ϕ^i, $i = 1, 2, \cdots, p$ with two-point functions normalized as follows

$$\langle \phi^i(z)\phi^j(w)\rangle = \langle \bar\phi^i(z)\bar\phi^j(w)\rangle = 0 \ , \ \langle \phi^i(z)\bar\phi^j(w)\rangle = -\delta^{ij}\log(z-w) \ . \tag{4.12}$$

Thanks to the linear structure of the commutation relations (3.1) the tower of higher spin fields

$$W^s(z) = B(s) \sum_{i=1}^{p} \sum_{k=1}^{s-1} (-1)^k A^s_k \partial^k_z \phi^i \partial^{s-k}_z \bar\phi^i \tag{4.13}$$

with A^s_k and $B(s)$ given as before, provides a bosonic realization of (3.1) with $c = 2p$.

This realization of W_∞ arises naturally in the CFT models described by the Grassmannian cosets $G_\infty(p)$. Notice that for all p, the $SU(p)_N$ current algebra abelianizes in the limit $N \to \infty$ and becomes a $U(1)^{p^2-1}$ current algebra. Therefore, the Grassmannian coset models $G_\infty(p)$ are parametrized by $U(1)^{2p}$ affine currents, which can be identified with the fields $\partial\phi^i$, $\partial\bar\phi^i$ ($i = 1, 2, \cdots, p$) in (4.12) , in exact analogy with the $p = 1$ case. W_∞ is a subalgebra of the parafermionic algebra of these models or alternatively it is a subalgebra of the enveloping algebra of the $U(1)^{2p}$ current algebra.

We conclude this section by considering a real counterpart of the (complex) W_∞ algebra, which we denote by RW_∞. This algebra is obtained from the real coset models $O(N)_1 \otimes O(N)_p / O(N)_{p+1}$ and has $c = p$ at large N. For $p = 1$, we have $c = 1$ for all values of N and the corresponding algebra is an enlargement of the chiral algebra of rational Z_2 orbifold models with $c = 1$. It is known, [14], that these orbifold models have a chiral algebra generated by Virasoro primaries belonging in the U(1) current module, with spins

$s = n^2$, $n = 1, 2, \cdots$ and a pair of vertex operators with spin N, $exp(\pm i\sqrt{2N}\phi)$, where, if the radius of compactification is written as $R^2 = 2p/p'$ with $(p, p') = 1$, then $N = pp'$. As $N \to \infty$, the latter operators decouple and only the current module remains. In the orbifold model, all operators containing an odd number of $U(1)$ currents are projected out. If we enlarge the set by including also quasiprimary operators we will obtain RW_∞, which is generated in terms of a single real scalar field by the operators

$$W^{2s}(z) = B(2s) \sum_{k=1}^{2s-1} (-1)^k A_k^{2s} \partial_z^k \phi \partial_z^{2s-k} \phi \ . \tag{4.14}$$

Hence, RW_∞ is the quotient of W_∞ by the ideal of fields with odd spins. This algebra was denoted by $W_{\frac{\infty}{2}}$ in the work of PRS. There is an alternative real version of W_∞ which also contains an abelian current, [15]. This algebra, denoted by $W_{1+\infty}$, can be realized in the U(1) current module generated by a single real scalar field or equivalently by a complex fermion, [16].

5. Unitary representations of W_∞

Highest weight representations of W-algebras can be constructed in analogy with the Virasoro algebra. We start with the standard $SL(2, C)$ invariant vacuum which is also invariant under the action of the W-algebra, ie.

$$W_n^s |0\rangle = 0 \ , \ \ n \geq 1 - s \ . \tag{5.1}$$

A hw state $|Q\rangle$ is one that is annihilated by the raising operators,

$$W_{n>0}^s |Q\rangle = 0 \ , \ \ W_0^s |Q\rangle = Q_s |Q\rangle \ . \tag{5.2}$$

Then, the W-representation is built on the hw state by the action of the lowering operators, $W_{n<0}^s$. Hw states are created from the vacuum by local operators $V_Q(z)$, $V_Q(0)|0\rangle = |Q\rangle$. The hw conditions (5.2) translate into the following OPE for the primary operators $V_Q(z)$

$$W^s(z) V_Q(w) = Q_s \frac{V_Q(w)}{(z-w)^s} + \mathcal{O}[(z-w)^{1-s}] \ , \tag{5.3}$$

where $\Delta \equiv Q_2$ is the (left) scaling dimension. These definitions are applicable to any W-algebra and hence to W_∞ as well.

Unitarity constraints the values of the central charge of W_N minimal models to be in the discrete series (3.7), [4]. In this case, using the hermiticity condition

$$(W_n^s)^\dagger = W_{-n}^s \tag{5.4}$$

we find that there are no negative norm states in the theory. Further analysis has shown that the spectrum of dimensions of W_N-primary fields, is given by the formula

$$\Delta(\{k_i\};\{k_i'\}) = \frac{12\left[\sum_{i=1}^{N-1}\left((N+p)k_i-(N+p+1)k_i'\right)\vec{\omega}_i\right]^2 - N(N^2-1)}{24(N+p)(N+p+1)} \ , \tag{5.5}$$

where $\{k_i\}$ and $\{k_i'\}$ are sets of positive integers subject to the constraints

$$\sum_{i=1}^{N-1} k_i \leq N+p \ , \ \sum_{i=1}^{N-1} k_i' \leq N+p-1 \tag{5.6}$$

and $\vec{\omega}_i$ are the fundamental weights of $SU(N)$, which satisfy the conditions

$$\vec{\omega}_i \cdot \vec{\omega}_j = \frac{i(N-j)}{N} \ , \ i \leq j \ . \tag{5.7}$$

It is clear that when $N \to \infty$, the sequence (3.7) has no upper bound and the unitary hw representations of W_∞ should occur with central charge $c = 2p$ only, with $p \in Z^+$. However, the number of conformal blocks becomes infinite in this limit and the concept of minimality is not very practical for solving these models. Moreover, as can be readily seen from eq. (5.5), the spectrum of dimensions collapses into three disjoint sets for large N. The first class denoted by C_0 encompasses the operators whose dimension approaches zero. The second, C_F, contains the operators that reach finite scaling dimension, while the third, C_∞, contains the operators that have unbounded dimension as $N \to \infty$. Here, we mainly deal with C_F and will comment briefly on C_0. The class C_∞ deserves a separate study and we suspect that there may be non-trivial structure hidden in it.

It is obvious from sections 3 and 4 that the W_∞ algebra we constructed with $c = 2p$ is a subalgebra of the enveloping algebra of the $U(1)^{2p}$ current algebra. The W-generators are bilinear in the current modes (cf. eq. (4.13)) and certainly satisfy the hermiticity condition (5.4). Thus, hw irreducible representations of the $U(1)^{2p}$ current algebra decompose into representations of the W_∞ algebra (3.1) with $c = 2p$. This way we obtain a very simple class of hw unitary irreducible representations of W_∞. However, it is not clear whether the list of all hw unitary representations is exhausted by our construction.

Let us consider first the simplest case with $c = 2$, which corresponds to Z_∞ parafermions. For simplicity, as we explained earlier, it is more convenient to use the large (level) N limit of the $SU(2)$ WZW model and then translate the results to the parafermionic coset (3.8) with $p = 1$. Let $[j]$ be a hw representation of the $SU(2)$ current algebra with spin j which contains at the top level $2j+1$ states labeled by m, $-j \leq m \leq j$, all with dimension $\Delta_j = j(j+1)/(N+2)$. These states give rise to hw states of the W_N algebra. In particular, the $m = j$ state gives rise to the parafermionic spin field σ_j, [12]. In the $SU(2)$ WZW model, we have also three sectors of operators as $N \to \infty$. $\tilde{C}_0$ contains primary fields with finite (fixed) j and dimension $\Delta_j \sim 1/N$, $\tilde{C}_F$ contains operators with spin $j \sim \sqrt{N}$ and finite dimension and $\tilde{C}_\infty$ contains operators with spin $j \sim N$ and unbounded dimension. The operators in $\tilde{C}_0$ form a closed OPE, which is nothing else but the Clebsch-Gordan algebra of ordinary $SU(2)$ representations. In particular, the OPE coefficients calculated in [17] reduce to the $SU(2)$ Clebsch-Gordan coefficients.

For the operators in $\tilde{C}_F$ the situation is more interesting. Let us parametrize them using their dimension $\Delta = \lim_{j,N\to\infty} \frac{j(j+1)}{N+2}$, which is finite provided that $j/\sqrt{N}$ is kept fixed. It is obvious that any (non-negative) real value for Δ can be obtained this way. Now, if we analyse the OPE coefficients in this regime we will find that their three-point couplings (OPE coefficients) will be exponentially suppressed as $e^{-b\sqrt{N}}$ and thus vanish as $N \to \infty$, unless

$$\sqrt{\Delta_1} = \sqrt{\Delta_2} + \sqrt{\Delta_3} \tag{5.8}$$

or cyclic permutations. This is precisely the composition law for vertex operators. Since the $SU(2)$ currents in the large N limit become abelian, the primary fields in this sector must be of the form $exp[i\alpha_3\rho + i\alpha\phi + i\bar{\alpha}\bar{\phi}]$, in the notation of section 3. The W_∞ primary fields can be obtained by factoring out the ρ-field dependence and therefore assume the form

$$V_{\alpha,\bar{\alpha}}(z) =: exp(i\alpha\phi(z) + i\bar{\alpha}\bar{\phi}(z)) : \quad . \tag{5.9}$$

Explicit calculation shows that the primary operators (5.9) have W_∞ charges given by

$$Q_s^{\alpha,\bar{\alpha}} = (1 + (-1)^s)\frac{2^{s-3}(s-1)!(s-2)!}{(2s-3)!!}\Delta \ , \quad \Delta = |a|^2 \ . \tag{5.10}$$

We note that these operators have zero charges under the W-generators with odd s. This does not mean however that the odd part of the algebra acts trivially on the representation module. The rest of the representation is obtained (in a standard way) by acting with the lowering operators, $W^s_{n<0}$. However, it is not clear up to this point, into how many irreducible W_∞ representations a single affine $U(1)$ representation decomposes. To answer the question we study the character formulae for these representations.

Recall that the character of a spin j affine $SU(2)$ representation is given by, [18]

$$\chi_j(\tau, z) \equiv Tr_j\left[e^{2\pi i(\tau(L_0 - \frac{c}{24}) + zJ_0^3)}\right] = i\frac{\vartheta_{2j+1,N+2}(z|\tau) - \vartheta_{-2j-1,N+2}(z|\tau)}{\vartheta_1(z|\tau)} \quad , \tag{5.11}$$

where

$$\vartheta_{m,N}(z|\tau) = \sum_{n\in Z+\frac{m}{2N}} e^{2\pi i N(n^2\tau - nz)} \tag{5.12}$$

and $\vartheta_1(z|\tau)$ is the standard Jacobi ϑ-function. The reduced characters are obtained from eq. (5.11) by setting $z = 0$ and provide the building blocks for the $SU(2)$ WZW partition function in the absence of background fields. The corresponding characters for the hw representations of the W_N algebra can be written in terms of the string functions c^{2j}_m, [19] which are defined implicitly through, [18]

$$\chi_j(\tau,z)=\sum_{m=1-N}^{N} c^{2j}_m(\tau)\vartheta_{m,N}(z|\tau) \ . \tag{5.13}$$

The W_N characters are given by $\chi^j_m(\tau)=\eta(\tau)c^{2j}_m(\tau)$, where η is the Dedekind η-function. Now by taking the $N\to\infty$ limit for the class of $\tilde{C}_F$ representations, we find

$$\chi_j(\tau)=2\sqrt{\Delta N}\frac{e^{2\pi i\tau\Delta}}{\eta^3}\left[1+\mathcal{O}(\frac{1}{N})\right] \ , \tag{5.14}$$

which implies that in our case the W_∞ character is

$$\chi_a(q)\equiv Tr[q^{L_0-\frac{c}{24}}]=\frac{q^{|a|^2-\frac{1}{12}}}{\prod_{n=1}^{\infty}(1-q^n)^2} \ . \tag{5.15}$$

Thus, the W_∞ character at $c=2$ coincides with the $U(1)^2$ character, which proves that a $U(1)^2$ representation decomposes precisely into one W_∞ irreducible representation.

Hw representations of W_∞ with $c=2p$ are obtained from hw representations of the $U(1)^{2p}$ current algebra, in a similar way. The primary operators of the $U(1)^{2p}$ current algebra are generated by the vertex operators

$$V_{\vec{\alpha},\vec{\bar{\alpha}}}(z)=:exp[i\vec{\alpha}\vec{\phi}(z)+i\vec{\bar{\alpha}}\vec{\bar{\phi}}(z)]: \ , \tag{5.16}$$

using the p-component free scalar fields, $\vec{\phi}$ and $\vec{\bar{\phi}}$. They also generate hw unitary representations of W_∞ with $c=2p$. However, for $p>1$, the representation is completely reducible. A character analysis like the one for $p=1$ can reveal into how many irreducible representations a single $U(1)^{2p}$ representation decomposes. Such analysis, although straightforward in principle, is quite involved and we reserve an answer for the future. It is highly plausible that every unitary hw irreducible representation of W_∞ can be constructed from $U(1)^{2p}$ representations. However, we do not have a proof of this statement presently.

6. The algebra W^p_∞ and its large p limit

In ref. [3] we introduced a $U(p)$-matrix generalization of W_∞, denoted by W^p_∞, taking advantage of the additional symmetries that the Grassmannian coset models $G_\infty(p)$ possess. In this section we review the construction of W^p_∞ and its connection with the algebra of symplectic diffeomorphisms in four dimensions, as $p\to\infty$.

Let $\{T^\alpha\,;\alpha=0,1,\cdots,p^2-1\}$ be a basis in the Lie algebra of $U(p)\simeq SU(p)\otimes U(1)$,[§] so that in the fundamental representation T^0 coincides with the $p\times p$ unit matrix

[§]Greek indices range from 0 to p^2-1 whereas Latin ones from 1 to p^2-1.

and $\{T^a \ ; a = 1, 2, \cdots, p^2 - 1\}$ are traceless hermitian matrices which satisfy the $SU(p)$ commutation relations

$$[T^a, T^b] = f^{abc} T^c \ . \tag{6.1}$$

The summation convention over repeated $SU(p)$ indices is implicitly assumed. Moreover, we may always normalize the generators $\{T^a\}$ so that $Tr\left(T^a T^b\right) = p\delta^{ab}$, ie.

$$T^a T^b = \delta^{ab} 1_p + \frac{1}{2} f^{abc} T^c + d^{abc} T^c \ , \tag{6.2}$$

where

$$d^{abc} = \frac{1}{2p} Tr\left((T^a T^b + T^b T^a) T^c\right) \tag{6.3}$$

is the third order completely symmetric Casimir tensor (which is identically zero for SU(2)). Then, eq. (6.2) describes the decomposition of the matrix $T^a T^b$ into its trace, antisymmetric and traceless-symmetric parts.

Notice that the generating fields (4.13) of W_∞ with $c = 2p$ are of the form

$$W^s_\alpha(z) = B(s) \sum_{i,j=1}^{p} \sum_{k=1}^{s-1} (-1)^k A^s_k (T^\alpha)_{ij} \partial^k_z \phi^i \partial^{s-k}_z \bar{\phi}^j \tag{6.4}$$

with $\alpha = 0$. This motivates the introduction of the $U(p)$ fields $W^s_\alpha(z)$ for all $\alpha = 0, 1, \cdots, p^2 - 1$. In this setting, the operators (4.13) correspond to the $U(1)$ trace part of $U(p)$. The generalized higher spin fields (6.4) form a closed linear operator algebra which is a multicomponent extension of W_∞. Its structure is completely determined by the OPE

$$W^s_\alpha(z) W^{s'}_\beta(w) = B(s)B(s') \left[\frac{(2s-2)!}{s(s-1)} \frac{\delta^{s,s'} Tr(T^\alpha T^\beta)}{(z-w)^{s+s'}} + \sum_{l=1}^{s+s'-2} \frac{R^{ss'}_{\alpha\beta;l}(\partial_w \phi, \partial_w \bar{\phi})}{(z-w)^l} \right] \ , \tag{6.5}$$

where

$$R^{ss'}_{\alpha\beta;l}(\partial\phi, \partial\bar{\phi}) = \sum_{i,j=1}^{p} \sum_{k=1}^{s-1} \sum_{k'=1}^{s'-1} (-1)^{k'} A^s_k A^{s'}_{k'} \frac{(k+k'-1)!}{(k+k'-l)!} \cdot$$

$$\cdot \left[(-1)^s (T^\alpha T^\beta)_{ij} \partial^{s+k'-l} \phi^i \partial^{s'-k'} \bar{\phi}^j + (-1)^{s'} (T^\beta T^\alpha)_{ij} \partial^{s'-k'} \phi^i \partial^{s+k'-l} \bar{\phi}^j \right] \tag{6.6}$$

for all $s, s' \geq 2$, using the two-point functions (4.12). This generalizes the results (4.8) and (4.9) to a collection of p complex scalar fields.

Since T^0 is represented by the identity matrix, the OPE (6.5) yields

$$[W^s_{0,m}, W^{s'}_{\alpha,n}] = ((s'-1)m - (s-1)n) W^{s+s'-2}_{\alpha,m+n} + c_s(m) \delta^{\alpha,0} \delta^{s,s'} \delta_{m+n,0} +$$

$$+ g^{ss'}_2(m,n) W^{s+s'-4}_{\alpha,m+n} + g^{ss'}_4(m,n) W^{s+s'-6}_{\alpha,m+n} + \cdots \tag{6.7}$$

for $0 \leq \alpha \leq p^2 - 1$, in exact analogy with the calculation outlined in section 4. $c_s(m)$ is given by eq. (3.2) with $c = 2p$ and for $\alpha = 0$ the commutation relations (6.7) reproduce the PRS algebra, as required. The remaining commutation relations of W_∞^p for both $\alpha, \beta \neq 0$ are less trivial to derive from eqs. (6.5) and (6.6) and require a more careful analysis. We point out that the contributions of the trace and symmetric parts of T^aT^b to the commutation relations of the fields $W_a^s(z)$ for all a are analogous to eq. (6.7), provided that the indices a, b are displayed accordingly. However, the contribution of the antisymmetric piece $\sim f^{abc}$ flips the sign of the terms $(-1)^{s'}\partial^{s'-k'}\phi^i\partial^{s+k'-l}\bar{\phi}^j$ in eq. (6.6) and therefore the decomposition of the operators $R^{ss'}$ into linear combinations of the generating W-fields and their derivatives is different. Explicit calculation shows that in this case, the contribution of the antisymmetric part to the OPE (6.5) involves once again the W-fields and their derivatives linearly, but with spin $s+s'-1$, $s+s'-3, \cdots$ instead of $s+s'-2$, $s+s'-4, \cdots$. Also, the coefficients (structure constants) of these fields written in terms of Fourier modes, are given by the combinatorial expressions $g_l^{ss'}$ (cf. eq. (3.3)) with l extrapolated to odd integer values. Thus, the commutation relations are

$$[W_{a,m}^s, W_{b,n}^{s'}] = ((s'-1)m-(s-1)n)\left(\delta^{ab}W_{0,m+n}^{s+s'-2} + d^{abc}W_{c,m+n}^{s+s'-2}\right) +$$
$$+c_s(m)\delta^{ab}\delta^{s,s'}\delta_{m+n,0} + \sum_{r>0} g_{2r}^{ss'}(m,n)\left(\delta^{ab}W_{0,m+n}^{s+s'-2-2r} + d^{abc}W_{c,m+n}^{s+s'-2-2r}\right) -$$
$$-\frac{1}{2}f^{abc}\left[\frac{1}{2}W_{c,m+n}^{s+s'-1} + \sum_{r>0} g_{2r-1}^{ss'}(m,n)W_{c,m+n}^{s+s'-1-2r}\right] \tag{6.8}$$

with $c = 2p$.[¶] In this expression the summations over r terminate either with W^2 or W^3 depending on whether $s+s'$ is even or odd.

The infinite dimensional algebra W_∞^p described by (6.7) and (6.8) provides a $U(p)$-matrix generalization of W_∞. In our construction associativity is manifest, but it can also be verified directly using the Jacobi identities for W_∞^p. The main new feature of this algebra is the presence of fields with spin $s+s'-1-2r$ in the commutation relations (6.8). As a result, the spin-2 W_∞^p-fields W_α^2 do not form a closed subalgebra unless $\alpha = 0$. In particular, we have

$$[W_{0,m}^2, W_{\alpha,n}^2] = (m-n)W_{\alpha,m+n}^2 + \frac{p}{6}m(m^2-1)\delta^{\alpha,0}\delta_{m+n,0} \ , \tag{6.9}$$

$$[W_{a,m}^2, W_{b,n}^2] = -\frac{1}{4}f^{abc}W_{c,m+n}^3 + (m-n)(\delta^{ab}W_{0,m+n}^2 + d^{abc}W_{c,m+n}^2) +$$
$$+\frac{p}{6}m(m^2-1)\delta^{ab}\delta_{m+n,0} \ , \tag{6.10}$$

where $W_0^2(z)$ is identified with the stress tensor of the $G_\infty(p)$ models. The commutation relations (6.9) and (6.10) suggest that unlike the case of spin-1 fields, a Yang-Mills type generalization of spin-2 fields cannot be implemented consistently without the introduction of higher spin fields. This novelty is not shared by spin-1 fields because for them we

[¶]A proof of (6.8) can be obtained by a U(p)-coloring of the "lone-star" product of ref. [15].

simply have $s+s'-1=1$. It would be interesting to investigate further the geometrical meaning of colored spin-2 fields, not only in the present framework, but also in the context of gravitational theories in spacetime dimensions $d \geq 2$.

Next we study the structure of W_∞^p, when $p \to \infty$. For this purpose we need to know how the algebra $U(p)$ behaves in the large p limit. Recall that in the fundamental representation there exists a basis $T^{\vec{k}}$ in which the commutation relations of $U(p)$ take the form, [20]

$$[T^{\vec{k}}, T^{\vec{l}}] = -2i \sin\left[\frac{\pi}{p}(\vec{k}\times\vec{l})\right] T^{\vec{k}+\vec{l}} \tag{6.11}$$

and

$$[T^{\vec{k}}, T^{\vec{l}}]_+ = 2\cos\left[\frac{\pi}{p}(\vec{k}\times\vec{l})\right] T^{\vec{k}+\vec{l}} \ . \tag{6.12}$$

Here, $\vec{k}=(k_1,k_2)$ and $\vec{l}=(l_1,l_2)$ are two-dimensional vectors with non-negative integer entries and $T^{\vec{0}}=1_p$. This basis is defined by introducing

$$g = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \omega & 0 & \cdots & 0 \\ 0 & 0 & \omega^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \omega^{p-1} \end{pmatrix} \ , \ h = \sqrt{\omega} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -1 & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{6.13}$$

with $\omega = exp(2\pi i/p)$. Clearly, we have that

$$hg = \omega gh \ \ , \ \ g^p = h^p = 1_p \ . \tag{6.14}$$

The set of $p \times p$ matrices

$$T^{\vec{k}} \equiv \omega^{k_1 k_2/2} g^{k_1} h^{k_2} \tag{6.15}$$

with $0 \leq k_1, k_2 \leq p-1$ are independent and provide a complete basis for the Lie algebra of $U(p)$.‖ Then, it can be easily verified that

$$T^{\vec{k}} T^{\vec{l}} = \omega^{\vec{l}\times\vec{k}/2} T^{\vec{k}+\vec{l}} \ , \tag{6.16}$$

which is equivalent to the relations (6.11) and (6.12). Also, we may extend the range of $\vec{k}$ to negative integers by noting that

$$(T^{\vec{k}})^\dagger = T^{-\vec{k}} \ . \tag{6.17}$$

In this basis, the commutation relations (6.7) and (6.8) of W_∞^p become

$$[W_m^{s,\vec{k}}, W_n^{s',\vec{l}}] = ((s'-1)m - (s-1)n)\cos\left[\frac{\pi}{p}(\vec{k}\times\vec{l})\right] W_{m+n}^{s+s'-2,\vec{k}+\vec{l}} +$$

‖Strictly speaking, the previous representation is valid for p even. When p is odd the definitions are slightly different and amount to replacing π with 2π in (6.11) and (6.12). However, since we are eventually interested in the large p limit, this difference will not alter our conclusions provided that the rescaling we perform later incorporates the difference accordingly.

$$+c_s(m)\delta_{\vec{k}+\vec{l},0}\delta_{s,s'}\delta_{m+n,0} + \cos\left[\frac{\pi}{p}(\vec{k}\times\vec{l})\right]\sum_{r>0} g_{2r}^{ss'}(m,n)W_{m+n}^{s+s'-2-2r,\vec{k}+\vec{l}}+$$

$$+i\sin\left[\frac{\pi}{p}(\vec{k}\times\vec{l})\right]\left[\frac{1}{2}W_{m+n}^{s+s'-1,\vec{k}+\vec{l}} + \sum_{r>0} g_{2r-1}^{ss'}(m,n)W_{m+n}^{s+s'-1-2r,\vec{k}+\vec{l}}\right] \quad . \tag{6.18}$$

Since $c = 2p$, the central terms in (6.18) diverge linearly as $p \to \infty$. It is worth mentioning at this point that the large p limit of $U(p)$ (and hence W_∞^∞) might depend on the choice of basis. Different bases are related to each other linearly and for finite p, they provide equivalent descriptions of $U(p)$. However, as $p \to \infty$, the change of basis might involve infinite sums (which could possibly not converge) and the resulting commutation relations of $U(\infty)$ would be equivalent only formally.

Let us consider a classical version of (6.18) which means dropping the central terms. If we renormalize the generators as follows

$$\tilde{W}_m^{s,\vec{k}} = \left(\frac{i\pi}{2p}\right)^{s-2} W_m^{s,\vec{k}} \quad , \tag{6.19}$$

in the limit $p \to \infty$, eq. (6.18) will become

$$[\tilde{W}_m^{s,\vec{k}}, \tilde{W}_n^{s',\vec{l}}] = ((s'-1)m - (s-1)n)\tilde{W}_{m+n}^{s+s'-2,\vec{k}+\vec{l}} + (\vec{k}\times\vec{l})\tilde{W}_{m+n}^{s+s'-1,\vec{k}+\vec{l}} \quad . \tag{6.20}$$

It is rather easy to show that the algebra (6.20) describes symplectic diffeomorphisms in four dimensions. To demonstrate this explicitly we have to choose a specific basis. Let us consider a 4-d phase space with (local) coordinates x_1, x_2 and respective momenta p_1, p_2. The classical Poisson bracket is defined as usual

$$\{F_1(\vec{x},\vec{p}), F_2(\vec{x},\vec{p})\} = \sum_{i=1}^{2}\left[\frac{\partial F_1}{\partial x_i}\frac{\partial F_2}{\partial p_i} - \frac{\partial F_1}{\partial p_i}\frac{\partial F_2}{\partial x_i}\right] \tag{6.21}$$

and yields the commutation relations (6.20) using a basis of functions

$$F_m^{s,\vec{k}} = x_1^{m+s-1}p_1^{s-1}e^{ik_2x_2}e^{ik_1p_2} \quad . \tag{6.22}$$

Notice that if we had kept the central terms in eq. (6.18), then in the large p limit, under the rescaling (6.19), we would have obtained (6.20) again with a single central term surviving only in the commutation relations of the spin-2 fields. However, this central term is still linearly divergent and further work is required to regularize it and elucidate its meaning in quantum field theory.

Finally, the infinite dimensional subalgebra of (6.18) generated by the zero modes $m = n = 0$, can be identified with the loop algebra of area preserving diffeomorphisms in the limit $p \to \infty$. For this we consider (6.20) and let $F^{r,\vec{k}} = \tilde{W}_0^{r+1,\vec{k}}$. Then, we obtain the commutation relations

$$[F^{r,\vec{k}}, F^{r',\vec{l}}] = (\vec{k}\times\vec{l})F^{r+r',\vec{k}+\vec{l}} \quad , \tag{6.23}$$

which describe the loop algebra of area preserving diffeomorphisms of a 2-torus, $S^1 \times S^1$.

7. Conclusions

In this review paper we have presented an account of the structure and representations of the W_∞ algebra and its $U(p)$-matrix generalization, W_∞^p. W_∞ is the large N limit of the W_N algebras of CFT. In particular, we were able to show that W_∞ coincides with the linear infinite dimensional algebra proposed by PRS, using a CFT realization in terms of free scalar fields. We have also obtained all its hw unitary irreducible representations which are limits of the corresponding representations of W_N algebras in CFT and exist for $c = 2p$, with $p \in Z^+$. We conjecture that the representations found here exhaust all such hw unitary irreducible representations with finite central charge. For $p > 1$, we have shown that there exists a non-abelian $U(p)$-matrix generalization of W_∞, denoted by W_∞^p, which is realized in the Grassmannian coset models $G_\infty(p)$ and contains W_∞ as a subalgebra. Hw representations for this algebra can also be constructed from hw representations of $U(1)^{2p}$ current algebra, along the same lines we described throughout this paper. By taking a suitable $p \to \infty$ limit of W_∞^p, we obtained the algebra of symplectic diffeomorphisms in four dimensions, which also contains as a subalgebra the loop algebra of area preserving diffeomorphisms of the torus.

There are many areas where the study of W_∞ and its generalizations may shed some light and/or provide a concrete organization framework. We conclude our presentation by listing some topics of current activity. W_∞ is intimately connected with the continual Toda equations (see for instance [1]), which describe 4-d self-dual Einstein equations on manifolds with a Killing symmetry. In this context, it is interesting to study the relations with the symmetry algebra of Penrose's twistor construction, [21, 22] and its quantum version through $N = 2$ string theory, [23].

W_∞ can be used to construct certain theories of 2-d quantum W-gravity. So far, there are two alternative definitions of W_N-gravity. In the topological field theory approach, one uses the moduli space of flat $SL(N)$ connections on Riemann surfaces, [24]. In the other (more conventional) approach, one writes directly an action by gauging the W_N algebra. It has been shown that the action of W_∞-gravity provides a master theory, which yields by truncation all W_N-gravity theories with $N \geq 2$, [25].

The W_∞ algebra might be relevant in the matrix approach to 2-d gravity as well, [26]. In the (N-1)-matrix models one finds that the correlation functions satisfy differential equations of the $sl(N)$-KdV hierarchy, [27], while the loop equations amount to certain W_N hw conditions on the τ-function associated with the second derivative of the partition function, [28]. It is highly plausible that W_∞ will play a pivotal role in the large N limit of multi-matrix models, which describe $D = 1$ string theory.

Last but not least, W_∞ and its generalizations seem to be related to infinite dimensional symmetries of off-critical integrable models and certain operator algebras in higher dimensions. It would be quite interesting to study all these problems in detail. A synthesis of the ideas we discussed in this paper might lead to a non-perturbative solution of $D = 1$ string theory in terms of 4-d (topological) gravity, as advocated by Witten, [24].

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