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Grassmannian Coset Models and Unitary Representations of W_∞ *

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Abstract

It is shown that the 2-d coset models $SU(p+1)_N/SU(p)_N \otimes U(1)$ provide unitary representations of the chiral operator algebra W_∞ in the large level ($N \to \infty$) limit, with central charge $c = 2p$. For $p \geq 2$, the corresponding field theories possess additional symmetries which give rise to a $U(p)$ matrix generalization of W_∞, denoted by W_∞^p. Its commutation relations are obtained in closed form for all values of p and W_∞ is identified with the $U(1)$ trace part of W_∞^p. It is also shown that W_∞^p at large p is associated with the algebra of symplectic diffeomorphisms in four dimensions.

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Recently there has been considerable interest in the structure of universal W-algebras which arise as large-N limits of the extended conformal symmetry algebras W_N generated by the stress tensor $T(z) = W^2(z)$ and a collection of additional conserved (chiral) fields $\{W^s(z); s = 3, 4, \cdots, N\}$ with spin s, [1-4]. Although the chiral operator algebra W_∞ might not be uniquely defined in the limiting procedure, its commutation relations capture the universal features of higher spin transformations in two dimensions. The determining relations of W_N are non-linear in general and for finite N they assume the form, written in terms of Fourier modes,

$$[W_m^s, W_n^{s'}] = \sum_{\{s_i\},\{k_i\}} C_{s_1 \cdots s_p}^{ss'}(m, n; k_1, \cdots, k_p; c) W_{k_1}^{s_1} W_{k_2}^{s_2} \cdots W_{k_p}^{s_p} \ , \qquad (1)$$

where c is the central charge of the Virasoro subalgebra, $s_1 + s_2 + \cdots s_p \leq s + s' - 2$, $k_1 + k_2 + \cdots + k_p = m + n$ and $W_n^0 = \delta_{n,0}$ (inclusion of the identity operator). For any given pair of spins (s, s'), the structure constants $C_{\{s_i\}}^{ss'}$ are not universal, in the sense that many of them depend implicitly on N, and this makes their computation highly non-trivial. However, taking a suitable limit in which $N \to \infty$, the structure of W-algebras simplifies considerably and the commutation relations of the resulting infinite dimensional symmetry algebra (when appropriately defined) are determined only by universal constants.

It has been established that W_∞ is closely related with the algebra of area preserving diffeomorphisms of 2-manifolds (plane or cylinder) whose commutation relations are

$$[W_m^s, W_n^{s'}] = ((s' - 1)m - (s - 1)n) W_{m+n}^{s+s'-2} \ , \qquad (2)$$

with both $s, s' \geq 2$ and $m, n \in Z$. To be more precise, W_∞ can be described as a deformation of the Lie algebra (2) using (non-trivial) cocycle terms which are local functionals of the generating W-fields with spin less than $s + s' - 2$, [2-4]. The existence of consistent gauge interactions among higher spin fields with all integer values of $s \geq 2$, imposes physical restrictions on the form of the deformation terms that differentiate W_∞ from the algebra of area preserving diffeomorphisms. For physical reasons, it is natural to expect that these terms are central or linear, but not quadratic (or higher polynomial) in the W-fields.

A 2-parameter deformation of this type was constructed recently by Pope, Romans and Shen (PRS), which up to redefinitions and renormalizations seems to be the most general available, [4]. To describe it more explicitly, we introduce the combinatorial expressions

$$g_l^{ss'}(m, n) = \frac{1}{2(l+1)!} \varphi_l^{ss'} N_l^{ss'}(m, n) \ , \qquad (3)$$

where,

$$\varphi_l^{ss'} = \sum_{k \geq 0} \frac{(-\frac{1}{2})_k (\frac{3}{2})_k (-\frac{l+1}{2})_k (-\frac{l}{2})_k}{k! (-s + \frac{3}{2})_k (-s' + \frac{3}{2})_k (s + s' - l - \frac{3}{2})_k} \ , \qquad (4)$$

$$N_l^{ss'}(m,n) = \sum_{k=0}^{l+1} (-1)^k \binom{l+1}{k} (2s-l-2)_k [2s'-k-2]_{l+1-k} \cdot$$
$$\cdot [s-1+m]_{l+1-k} [s'-1+n]_k \tag{5}$$

and

$$(a)_k \equiv a(a+1)(a+2)\cdots(a+k-1) \ , \tag{6a}$$

$$[a]_k \equiv a(a-1)(a-2)\cdots(a-k+1) \ . \tag{6b}$$

We also set $(a)_0 = [a]_0 = 1$ for all values of a. Then the commutation relations of the PRS W_∞ algebra are

$$[W_m^s, W_n^{s'}] = ((s'-1)m - (s-1)n) W_{m+n}^{s+s'-2} + q^{2(s-2)} c_s(m) \delta_{s,s'} \delta_{m+n,0} +$$
$$+ q^2 g_2^{ss'}(m,n) W_{m+n}^{s+s'-4} + q^4 g_4^{ss'}(m,n) W_{m+n}^{s+s'-6} + \cdots \ , \tag{7}$$

where the coefficients of the central terms are

$$c_s(m) = \frac{c}{2} m(m^2-1)(m^2-4)\cdots(m^2-(s-1)^2) \frac{2^{2(s-3)} s!(s-2)!}{(2s-1)!!(2s-3)!!} \tag{8}$$

and the sequence of $\cdots$ terms terminates with W_{m+n}^2 for $s+s'$ even and with W_{m+n}^3 for $s+s'$ odd. Setting $q=0$, we end up with a central term only in the Virasoro subalgebra of W_∞*. For $q \neq 0$ we may rescale the generators W_n^s by q^{s-2} and normalize the value of the q-parameter to 1. ¿From now on we choose to work with $q=1$ without loss of generality.

In [5] we constructed a field theoretic representation of the PRS algebra, using a complex free boson in two dimensions. In particular we found that the quasiprimary fields

$$W^s(z) = B(s) \sum_{k=1}^{s-1} (-1)^k A_k^s : \partial_z^k \phi \partial_z^{s-k} \bar{\phi} : \ , \tag{9}$$

which are bilinear in the $U(1) \otimes U(1)$ currents $\partial\phi$, $\partial\bar{\phi}$ (and their derivatives) provide a realization of the universal algebra (7) when

$$A_k^s = \frac{1}{s-1} \binom{s-1}{k} \binom{s-1}{s-k} \ , \ B(s) = \frac{2^{s-3} s!}{(2s-3)!!} \ . \tag{10}$$

This realization has central charge $c=2$ and arises naturally in the theory of Z_∞ parafermions which we described using the coset model $SU(2)_N/U(1)$ in the large level ($N \to \infty$) limit.

In fact, as we will demonstrate next, the theory $SU(2)_\infty/U(1)$ generates the simplest unitary representation of the chiral operator algebra W_N in the limit $N \to \infty$. As is well

*It can be verified directly that this is the only possible central extension of the area preserving diffeomorphism algebra (2), which is consistent with the Jacobi identities.

known, the unitary representations of W_∞ have central charge $c = 2p = 2, 4, 6, \cdots$, [1,5] and they can be described by the two dimensional coset models

$$\frac{SU(N)_1 \otimes SU(N)_p}{SU(N)_{p+1}} \tag{11}$$

in the limit $N \to \infty$. We will adopt a different picture here in order to avoid unnecessary complications dealing with the large N limit of $SU(N)$ and approach the unitary representations of the universal algebra (7) with $c = 2p$, using the Grassmannian coset models

$$G_N(p) = \frac{SU(p+1)_N}{SU(p)_N \otimes U(1)} \tag{12}$$

in the large level ($N \to \infty$) limit. The cosets (11) and (12) have the same central charge, the same chiral algebra and the same irreducible representations and therefore provide two equivalent descriptions of the minimal models of the W_N algebras for all $N \geq 2$. Clearly, for $p = 1$ the present discussion produces the results we have already derived in [5]. However, for $p \geq 2$ the situation becomes more interesting because as we will see later, there are $U(p)$-matrix generalizations of W_∞ associated with the existence of infinitely many additional symmetries in the Grassmannian coset models we consider.

Let us begin with the simple observation that once the bosonic realization (9) of the PRS algebra has been constructed, the generalization to arbitrary values $c = 2p$ with $p = 1, 2, 3, \cdots$ follows immediately, thanks to the linear structure of the commutation relations (7). For this we introduce p independent complex scalar fields ϕ^i, $i = 1, 2, \cdots, p$ with two-point functions normalized as follows

$$\langle \phi^i(z)\phi^j(w)\rangle = \langle \bar{\phi}^i(z)\bar{\phi}^j(w)\rangle = 0 \ , \ \langle \phi^i(z)\bar{\phi}^j(w)\rangle = -\delta^{ij}\log(z-w) \ . \tag{13}$$

The standard stress tensor of the theory is $W^2(z) \equiv T(z) = -\sum_{i=1}^{p} :\partial_z\phi^i\partial_z\bar{\phi}^i:$ and the tower of spin fields

$$W^s(z) = B(s)\sum_{i=1}^{p}\sum_{k=1}^{s-1}(-1)^k A_k^s :\partial_z^k\phi^i\partial_z^{s-k}\bar{\phi}^i: \tag{14}$$

with $s \geq 2$, provides a multicomponent generalization of (9) that yields a bosonic realization of the PRS algebra with $c = 2p$.

This representation arises naturally in the conformal field theory models described by $G_\infty(p)$. Notice that for all p, the $SU(p)_N$ current algebra flattens out in the limit $N \to \infty$ and becomes a $U(1)^{p^2-1}$ current algebra. Therefore, the Grassmannian coset models $G_\infty(p)$ are parametrized by $U(1)^p \otimes U(1)^p$ affine currents, which can be identified with the fields $\partial\phi^i$ and $\partial\bar{\phi}^i$ ($i = 1, 2, \cdots, p$) in (13). W_∞ is a subalgebra of the parafermionic algebra of these models or alternatively it is a subalgebra of the enveloping algebra of the $U(1)^{2p}$ current algebra. As $N \to \infty$, the spectrum of the $G_N(p)$ models behaves the same way as in the $p = 1$ case. In particular, the set of primary operators with finite dimension can be described as vertex operators in terms of the scalar fields $\vec{\phi}$ and $\vec{\bar{\phi}}$, ie. $V_{\vec{a},\vec{b}} = exp[i\vec{a}\cdot\vec{\phi} + i\vec{b}\cdot\vec{\bar{\phi}}]$. It is a trivial exercise to show that they generate highest weight unitary irreducible representations of the W_∞ algebra with $c = 2p$, in analogy with the results obtained in [5]. However what ceases to be true for $p \geq 2$ is that each $U(1)^{2p}$ highest weight representation provides a single unitary irreducible representation of W_∞. For arbitrary p, unitary representations of $U(1)^{2p}$ decompose into more than one irreducible representations of the W_∞ algebra with $c = 2p$. We intent to elaborate on this point later.

Before we proceed any further, it is worth stressing that the bosonic theory we obtain in the large-N limit of the coset models (12) is not uniquely defined. Strictly speaking, $G_\infty(p)$ does not correspond to a single conformal field theory, but to a multiparameter collection thereof. It is certainly true that in the limit $N \to \infty$ one obtains toroidal scalar field models characterized by certain parameters (metric and antisymmetric tensor). However, the exact values of these parameters depend on the limiting procedure, which determines the limiting values of the dimensions of primary fields. It seems plausible that if we take into account the extra parameters that characterize the target manifold, $G_\infty(p)$ will be described either as a family of 2-d models with toroidal moduli, or equivalently as a single quantum field theory in higher dimensions. We reserve further comments on this problem to a future publication.

Next we take advantage of the additional symmetries that the Grassmannian coset models $G_\infty(p)$ possess and construct a $U(p)$-matrix generalization of W_∞, denoted by W_∞^p. For this purpose we introduce a basis $\{X^\alpha; \alpha = 0, 1, 2, \cdots, p^2-1\}$ in the Lie algebra of the unitary group $U(p) \simeq SU(p) \otimes U(1)$, so that in the fundamental representation X^0 coincides with the $p \times p$ unit matrix and $\{X^a; a = 1, 2, \cdots, p^2-1\}$ are traceless hermitian matrices that satisfy the $SU(p)$ commutation relations

$$[X^a, X^b] = f^{abc} X^c \ . \tag{15}$$

The summation convention over repeated $SU(p)$ indices is implicitly assumed. Moreover, we may always choose (see for instance, [6]) $\{X^a\}$ in a way that $Tr(X^a X^b) = p\delta^{ab}$, ie.

$$X^a X^b = \delta^{ab} 1_p + \frac{1}{2} f^{abc} X^c + d^{abc} X^c \ , \tag{16}$$

where,

$$d^{abc} = \frac{1}{2p} Tr((X^a X^b + X^b X^a) X^c) \tag{17}$$

is the third order completely symmetric Casimir tensor (which is zero for SU(2)). Notice that the generating fields (14) of the universal W-algebra (7) with $c = 2p$ are of the form

$$W_\alpha^s(z) = B(s) \sum_{i,j=1}^{p} \sum_{k=1}^{s-1} (-1)^k A_k^s (X^\alpha)_{ij} : \partial_z^k \phi^i \partial_z^{s-k} \bar{\phi}^j : \tag{18}$$

with $\alpha = 0$. This motivates the introduction of the $U(p)$ fields $W_\alpha^s(z)$ for all $\alpha = 0, 1, 2, \cdots, p^2 - 1$. In this setting, the operators (14) correspond to the $U(1)$ trace part of $U(p)$. The generalized higher spin fields (18) form a closed linear operator algebra which is a multicomponent extension of W_∞. Our task is to determine its structure completely.

It is quite straightforward to obtain the operator product expansion (OPE),

$$W_\alpha^s(z) W_\beta^{s'}(w) = \frac{2^{3s-7} s!(s-1)!(s-2)!}{(2s-3)!!} \frac{\delta_{s,s'} Tr(X^\alpha X^\beta)}{(z-w)^{s+s'}} + \\ + B(s)B(s') \sum_{l=1}^{s+s'-2} \frac{R_{\alpha\beta;l}^{ss'}(\partial\phi, \partial\bar{\phi})}{(z-w)^l} \ , \quad (19)$$

where,

$$R_{\alpha\beta;l}^{ss'}(\partial\phi \, , \partial\bar{\phi}) = \sum_{i,j=1}^{p} \sum_{k=1}^{s-1} \sum_{k'=1}^{s'-1} (-1)^{k'} A_k^s A_{k'}^{s'} \frac{(k+k'-1)!}{(k+k'-l)!} \cdot \\ \cdot [(-1)^s (X^\alpha X^\beta)_{ij} \partial_w^{s+k'-l} \phi^i \partial_w^{s'-k'} \bar{\phi}^j + (-1)^{s'} (X^\beta X^\alpha)_{ij} \partial_w^{s'-k'} \phi^i \partial_w^{s+k'-l} \bar{\phi}^j] \quad (20)$$

for all $s, s' \geq 2$, using the two-point functions (13). This generalizes the results obtained in [5] to a collection of scalar fields $\phi_1, \phi_2, \cdots, \phi_p$ and their complex conjugates.

Since the generator X^0 is represented by the identity matrix, the OPE (19) yields immediately

$$[W_{0,m}^s, W_{\alpha,n}^{s'}] = ((s'-1)m - (s-1)n) W_{\alpha,m+n}^{s+s'-2} + c_s(m) \delta^{\alpha,0} \delta_{s,s'} \delta_{m+n,0} + \\ + g_2^{ss'}(m,n) W_{\alpha,m+n}^{s+s'-4} + g_4^{ss'}(m,n) W_{\alpha,m+n}^{s+s'-6} + \cdots \quad (21)$$

for all $\alpha = 0, 1, 2, \cdots, p^2 - 1$, in exact analogy with the calculation we performed in [5]. Here, $c_s(m)$ is given by (8) with $c = 2p$. As required, for $\alpha = 0$, the commutation relations (21) reproduce the PRS algebra (7).

The remaining commutation relations for both $\alpha, \beta \neq 0$ are less trivial to derive from (19) and (20) and require more careful analysis. The new element that complicates the situation here originates from the presence of the $SU(p)$ structure constants f^{abc}. To appreciate the significance of this point when it comes to the essential details of the calculations, recall that in the fundamental representation of $SU(p)$, (16) describes the decomposition of the matrix $X^a X^b$ into its trace, antisymmetric and traceless-symmetric parts. The contributions of the trace and symmetric parts to the commutation relations of the spin fields $W_a^s(z)$ for all a are exactly the same as in (21), provided that the indices a, b, c are displayed accordingly. However, the contribution of the antisymmetric part $\sim f^{abc}$ flips the sign of the terms $(-1)^{s'} \partial^{s'-k'} \phi^i \partial^{s+k'-l} \bar{\phi}^j$ in (20) and therefore the decomposition of the operators $R^{ss'}$ into linear combinations of the generating W-fields $W^{s+s'-2}$, $W^{s+s'-4}$, $W^{s+s'-6}$, $\cdots$ and their derivatives that we used to derive (21) is not applicable anymore. For this reason, the commutation relations of the fields $W_a^s(z)$ will not be of the (standard) PRS type (7) by coloring "blindly" the generators.

Explicit calculations have shown that in this case, the contribution of the antisymmetric part $\sim f^{abc}$ to the OPE (19) involves once again W-fields and their derivatives linearly, but with spin $s+s'-1,\ s+s'-3,\ \cdots$. Perhaps more amazing is the fact that the coefficients (structure constants) of these fields, written in terms of Fourier modes, are given by the combinatorial expressions $g_l^{ss'}$ (cf (3)) with l extrapolated in the range of odd integers. In particular we have verified extensively the following commutation relations,

$$[W^s_{a,m}\,,W^{s'}_{b,n}] = ((s'-1)m-(s-1)n)(\delta^{a,b}W^{s+s'-2}_{0,m+n}+d^{abc}W^{s+s'-2}_{c,m+n})+$$
$$+c_s(m)\delta^{a,b}\delta_{s,s'}\delta_{m+n,0}+\sum_{r\geq 1}g^{ss'}_{2r}(m,n)(\delta^{a,b}W^{s+s'-2-2r}_{0,m+n}+d^{abc}W^{s+s'-2-2r}_{c,m+n})-$$
$$-\frac{1}{2}f^{abc}[\frac{1}{2}W^{s+s'-1}_{c,m+n}+\sum_{r\geq 1}g^{ss'}_{2r-1}(m,n)W^{s+s'-1-2r}_{c,m+n}] \qquad (22)$$

with $c=2p$. In this expression the summations over r terminate either with W^2 or W^3 depending on whether $s+s'$ is even or odd.

The infinite dimensional algebra described by (21) and (22) provides a $U(p)$-matrix generalization of W_∞ and as we have already pointed out, it arises as a symmetry algebra of the Grassmannian coset models $G_N(p)$ at large N. For obvious reasons we denote this algebra by W^p_∞. Although in our construction associativity is manifest, one may also verify directly the compatibility of W^p_∞ with the Jacobi identities. The main new feature of this algebra is the presence of fields with spin $s+s'-1-2r$ in the commutation relations (22). As a result, the spin-2 $U(p)$-fields, W^2_α, do not form a closed subalgebra of W^p_∞, unless $\alpha=0$. That is,

$$[W^2_{0,m},W^2_{\alpha,n}] = (m-n)W^2_{\alpha,m+n}+\frac{p}{6}m(m^2-1)\delta^{\alpha,0}\delta_{m+n,0}\ , \qquad (23a)$$

$$[W^2_{a,m},W^2_{b,n}] = -\frac{1}{4}f^{abc}W^3_{c,m+n}+(m-n)(\delta^{a,b}W^2_{0,m+n}+d^{abc}W^2_{c,m+n})+$$
$$+\frac{p}{6}m(m^2-1)\delta^{a,b}\delta_{m+n,0}. \qquad (23b)$$

The field W^2_0 is identified with the stress tensor of the $G_\infty(p)$ models.

At this point it is appropriate to return to the highest weight (hw) irreducible representations of W_∞ with $c=2p$. It is obvious from (22) that W^2_a with $a=1,2,\cdots,p^2-1$ generate a hw state of W_∞, when acting on the vacuum. The remaining operators W^s_a are generated from the hw states by the action of the W_∞ algebra (cf. (21)). Thus W^p_∞ as a module, decomposes into W_∞ and p^2-1 hw irreducible representations thereof†. A similar thing happens with the $U(1)^{2p}$ representations generated by the vertex operators. They are reducible as W_∞ algebra representations. A character analysis like the one

†This is analogous to the way that the algebra (2), viewed as a module of the Virasoro algebra, decomposes into a direct sum of primary conformal fields with integer dimensions, [2].

performed in [5], although in principle possible, is very hard to do for general p. Thus although we know that the $U(1)^{2p}$ representations are reducible under W_∞ we cannot at the moment estimate into how many representations they decompose. It is highly plausible that every unitary irreducible representation of W_∞ can be constructed from representations of the current algebra $U(1)^{2p}$ for all p.

The commutation relations (23) suggest that unlike the case of spin-1 fields, a Yang-Mills type generalization of spin-2 fields cannot be implemented consistently without the introduction of higher spin fields. This novelty is not shared by spin-1 fields, because for them we simply have $s + s' - 1 = 1$. It would be interesting to investigate further the geometrical meaning of colored spin-2 fields not only in the present framework, but also in the context of gravitational theories in spacetime dimensions $D \geq 2$. It seems reasonable to expect that for a collection of massless spin-2 fields $W^2_\alpha(z)$, the classical notion of manifolds has to be generalized to "algebra manifolds" where tensor fields take their values in $U(p)$. However in view of the commutation relations (23), one has to fit into the picture transformations generated by higher spin fields as well. In this case, the results obtained in [7] for colored spin-2 fields need to be re-examined.

There are two natural questions one may ask in connection with the infinite dimensional algebra W^p_∞. First, is there an operator algebra W^p_N whose universal structure is described by W^p_∞, for $p \geq 2$? Second, what is the geometrical interpretation of W^p_∞ at large p? The answer to the first question is in the affirmative and in fact W^p_N coincides with the chiral algebra of the Grassmannian models (12). However, the corresponding commutation relations are quite complicated to write down in any detail. As far as the second question is concerned, we were able to derive some (partial) results for W^∞_∞ using a limiting procedure developed in [8]. Recall that in the fundamental representation the Lie algebra $U(p)$ can be parametrized by a set of (matrix) generators $X^{\vec{K}}$, where $\vec{K}$ is a two dimensional vector with non-negative integer entries and $X^{\vec{0}} = 1_p$. Although for finite p there are some periodicity conditions imposed on $\vec{K}$, these will be ignored here because we are interested only in the large p behaviour of the algebra. In this basis, the commutation relations of $U(p)$ assume the form,

$$[X^{\vec{K}}, X^{\vec{L}}] = -2i \sin[\frac{\pi}{p}(\vec{K} \times \vec{L})] X^{\vec{K}+\vec{L}} , \tag{24a}$$

$$[X^{\vec{K}}, X^{\vec{L}}]_+ = 2 \cos[\frac{\pi}{p}(\vec{K} \times \vec{L})] X^{\vec{K}+\vec{L}} . \tag{24b}$$

Strictly speaking, the formulae (24) are applicable to $U(p)$ only for p even, while for p odd one has to change π into 2π. However, this does not alter our conclusions provided that the rescalings we perform later incorporate the difference accordingly. Then, (21) and (22) become

$$[W^{s,\vec{K}}_m, W^{s',\vec{L}}_n] = ((s'-1)m - (s-1)n) \cos[\frac{\pi}{p}(\vec{K} \times \vec{L})] W^{s+s'-2,\vec{K}+\vec{L}}_{m+n} +$$

$$+c_s(m)\delta_{\vec{K}+\vec{L},\vec{0}}\delta_{m+n,0}\delta_{s,s'} + \sum_{r\geq 1} g_{2r}^{ss'}(m,n)\cos[\frac{\pi}{p}(\vec{K}\times\vec{L})]W_{m+n}^{s+s'-2-2r,\vec{K}+\vec{L}} +$$

$$+i\sin[\frac{\pi}{p}(\vec{K}\times\vec{L})][\frac{1}{2}W_{m+n}^{s+s'-1,\vec{K}+\vec{L}} + \sum_{r\geq 1} g_{2r-1}^{ss'}(m,n)W_{m+n}^{s+s'-1-2r,\vec{K}+\vec{L}}] \ . \qquad (25)$$

Since $c = 2p$, the central terms in (25) diverge linearly as $p \to \infty$.

Let us now consider a classical version of (25) which means dropping the central terms. If we renormalize the generators as follows

$$\tilde{W}_m^{s,\vec{K}} = (\frac{i\pi}{2p})^{s-2} W_m^{s,\vec{K}} \ , \qquad (26)$$

then as $p \to \infty$, (25) will become

$$[\tilde{W}_m^{s,\vec{K}}, \tilde{W}_n^{s',\vec{L}}]_{cl} = ((s'-1)m - (s-1)n)\tilde{W}_{m+n}^{s+s'-2,\vec{K}+\vec{L}} + (\vec{K}\times\vec{L})\tilde{W}_{m+n}^{s+s'-1,\vec{K}+\vec{L}} \ . \qquad (27)$$

It is rather easy to show that the algebra (27) is associated with the algebra of symplectic diffeomorphisms in four dimensions! To demonstrate this explicitly we have to choose a specific basis. Let us consider a four dimensional phase space with (local) coordinates x_1, x_2 and respective momenta p_1 and p_2. The classical Poisson bracket is defined as usual,

$$\{F_1(\vec{x},\vec{p}), F_2(\vec{x},\vec{p})\}_{PB} = \sum_{i=1}^{2}[\frac{\partial F_1}{\partial x_i}\frac{\partial F_2}{\partial p_i} - \frac{\partial F_1}{\partial p_i}\frac{\partial F_2}{\partial x_i}] \ . \qquad (28)$$

If we define a basis of functions in the 4-d phase space,

$$F_m^{s,\vec{K}} \equiv x_1^{m+s-1} p_1^{s-1} e^{ik_2 x_2} e^{ik_1 p_2} \ , \qquad (29)$$

where $\vec{K} \equiv (k_1, k_2)$, their Poisson bracket relations will coincide with (27). Notice that if we had kept the central terms in (25), then under the rescaling (26) we would have obtained (27) again with a single central term surviving only in the commutation relations of the spin-2 fields. However, this central term is still linearly divergent and further work is required to regularize it and elucidate its meaning.

In any event, W_∞^∞ seems to be closely related with the algebra of symplectic diffeomorphisms in four dimensions. As such it is a subalgebra of the full volume preserving diffeomorphism algebra. It is crucial to realize that in more that two (but even number) of dimensions, not all volume preserving diffeomorphisms arise as symplectic transformations. Of course the converse statement is obviously true always. In [9], W_∞ gravity was introduced and treated as a "light-cone" type gauge theory. It is expected that a covariant formulation of that theory would involve the algebra of symplectic diffeomorphisms in four dimensions or perhaps the full volume preserving diffeomorphism symmetry, [10]. It is intriguing to understand possible connections with the W_∞^∞ algebra constructed here. However, such relations are still mystery to us.

There is another interesting infinite dimensional algebra which can be obtained as a subalgebra of (25) in a certain limit. This is the loop algebra of area preserving diffeomorphisms. Its commutation relations are given by

$$[F^k_{\vec{m}}\, ,\, F^{k'}_{\vec{n}}] = (\vec{m}\times\vec{n})F^{k+k'}_{\vec{m}+\vec{n}}\ , \tag{30}$$

where $\vec{m}$, $\vec{n}$ are two dimensional vectors with integer entries and k, k' are integers. We will consider again a classical limit of (25) by dropping the central terms. Then the subalgebra of the zero modes is

$$[W_0^{s,\vec{K}}\, ,\, W_0^{s',\vec{L}}]_{cl} = \frac{i}{2}\sin[\frac{\pi}{p}(\vec{K}\times\vec{L})]W_0^{s+s'-1,\vec{K}+\vec{L}}\ . \tag{31}$$

Once again, we define new generators as follows

$$F^r_{\vec{K}} = \frac{2p}{i\pi}W_0^{r+1,\vec{K}} \tag{32}$$

and take the limit $p\to\infty$. As usual, we may extend the range of $\vec{K}$ to all integer values, [8]. Then, it is trivial to show that the operators $F^r_{\vec{K}}$ satisfy the commutation relations of the loop algebra of area preserving diffeomorphisms (30). This algebra arises as a hidden symmetry of the self-dual Einstein equations in four dimensions (see for instance, [11]). The latter have attracted considerable attention among particle physicists recently, after the realization that $N = 2$ string theory provides a quantization of the self-dual Einstein equations in four dimensions with space-time signature (2,2), [12]. It is plausible that W^∞_∞ and its (yet unknown) deformations are related to symmetries of the quantum theory of gravitational instantons as formulated through $N = 2$ string theory. We intent to investigate these problems in more detail.

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