

Fuchsian Differential Equations for Characters on the Torus: A Classification *

Elias B. Kiritsis

*Department of Physics
University of California
and
Theoretical Physics Group
Physics Division
Lawrence Berkeley Laboratory
1 Cyclotron Road
Berkeley, California 94720*

Abstract

Fuchsian Differential Equations for characters of Rational Conformal Field Theories on the torus are classified by invoking the solution of the Riemann- Hilbert problem and modular geometry. As a first application, all Rational Conformal Field Theories with two characters are classified.

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1. Introduction

Interest in two dimensional conformally invariant quantum field theories has been high in recent years, mainly because of the general interest in string theories. The subject has important applications in the theory of 2-d critical systems and classical string theory. A universality class of critical behavior in 2-d is described by a single Conformal Field Theory, (CFT). The set of all CFT's is also the set of all classical string ground states. CFT provided so far the widest set of exactly solvable quantum field theories.

A major problem in the subject is the classification, (and exact solution), of all CFT's. That would classify for example all classical ground states of string theory. There exist partial results like the classification of unitary CFT's with $c < 1$. Unfortunately the problem in its full generality seems very difficult. However there exist subsets of CFT's which are wide enough and where there is serious hope that the classification problem can be solved. The widest and most important such subset is that of Rational CFT's, (RCFT). There are various definitions of RCFT. In the language of [1], they are the theories that give rise to vector bundles on moduli space that have a finite rank. What the above means is that, for example, the number of holomorphic blocks needed to construct a correlation function is finite. In [2] it was proven that in a RCFT, c and the critical dimensions are rational numbers.

Most of the efforts so far in attacking the classification problem have been in using algebraic tools to study the consistency conditions on the data of RCFT, [3,4,5,6,7]. Notable exceptions are [2,8,9].

In this paper we will take a different approach to the problem by using algebro-analytic tools. In particular the theory of Fuchsian Differential Equations, (FDE's), seems to be very well suited to this kind of problem. It is also true that there exists a solid collection of mathematical results that makes life easier for a physicist.

The specific problem that will be addressed in this paper is the application of the Fuchsian theory to characters of RCFT's on the torus. It will also turn out that one-point functions on the torus as well as four-point functions on the sphere can be addressed at no extra expense. More general applications of the present framework will be discussed in a separate publication.

We will give a classification of FDE's satisfied by characters of RCFT's and discuss the constraints on their solutions stemming from the general principles of CFT. A key point to the classification above is the solution to the Riemann-Hilbert problem. As an application to the analysis above we will classify all theories with two holomorphic blocks (characters).*

*A preliminary step in this direction was taken in [9].

The structure of this paper is as follows. In section two we present some pertinent facts about the modular group, Γ and one of its subgroups, $\Gamma(2)$. In section three we discuss the constraints that the characters of RCFT must satisfy. In section four we give the general form of a FDE satisfied by characters subject to the constraints of modular covariance and certain other physical requirements. In section five we present the solution to the Riemann-Hilbert problem which we use to constraint further the FDE's to a countable set. In section six we make use of the previous results to give a complete classification of all RCFT's with two characters. In section seven we discuss how the results presented can be also applied with no modifications to the classification and determination of one-point functions on the torus and four-point functions on the sphere and with minor modifications to two-point functions on the torus. Section eight contains our conclusions and some directions for further work. In appendix A we give the explicit form of the n -th order FDE in the $\Gamma(2)$ invariant coordinate x and in appendix B we present formulae for characters of RCFT's when they reduce to algebraic functions.

2. The Modular Group and Modular Forms

The partition function of a CFT on the torus is naturally a function of the modulus of the torus, τ . The modulus is a complex number with a non-negative imaginary part. It determines the conformal class of the torus. The partition function must also be invariant under the global diffeomorphisms of the torus which form the group, $PSL(2, \mathbf{Z}) \equiv \Gamma$,

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad a, b, c, d \in \mathbf{Z}, ad - bc = 1 \quad (2.1)$$

Γ is generated by two elements, $S : \tau \rightarrow -\frac{1}{\tau}$ and $T : \tau \rightarrow \tau + 1$, which satisfy the relations $S^2 = (ST)^3 = 1$. Thus Γ is the free product of Z_2 and Z_3 , the former generated by S , the latter by ST . The standard fundamental region, H/Γ of the modular group is,

$$H/\Gamma = \left\{ \tau \in H : -\frac{1}{2} \leq \text{Re}\tau \leq \frac{1}{2}, |\tau| \geq 1 \right\} \quad (2.2)$$

where H is the upper-half plane. It is convenient to compactify H/Γ by adding the point $\tau = i\infty$. This corresponds to a torus pinched along a homology cycle. H/Γ^\dagger is a Riemann surface of genus zero. There are three distinguished points of H/Γ that require special attention. The point $\tau = i$ is a fixed point of Γ of order two. Thus H/Γ has an orbifold singularity there and if we go around that point once we pick an angle π instead of 2π . The point $\tau = \rho = e^{\frac{2\pi i}{3}}$ is a fixed point of Γ of order three and H/Γ has an orbifold singularity there of order three. The parabolic point $\tau = i\infty$ is an orbifold point of infinite order. Thus the proper coordinate around that point is $q = e^{2\pi i\tau}$.

[†]From now on by H/Γ we mean the compactified one.

A modular form f of dimension k is a form on H which under the action of the modular group is covariant,

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{-k} f(\tau), \quad k \in \mathbf{Z} \quad (2.3)$$

It is an obvious fact that there exist no modular forms of odd dimension. A simple form of the Riemann-Roch theorem that gives the number of zeros minus the number of poles has to be modified due to the presence of the orbifold singularities. Let $f(\tau)$ be a modular form of dimension $2k \in 2\mathbf{Z}$. Let $n_\infty, \frac{n_i}{2}, \frac{n_\rho}{3}$ be the order of zeros at the points $\tau = i\infty, i, \rho$ if n_∞, n_i, n_ρ are positive, or the number of poles at these points if these numbers are negative. Let also $N_0(0)$ be the number of zeros and $N_0(\infty)$ the number of poles anywhere else except the aforementioned points. Then, [10],

$$N_0(0) - N_0(\infty) = \frac{k}{6} - \frac{n_\rho}{3} - \frac{n_i}{2} - n_\infty \quad (2.4)$$

The ring of modular functions, (forms of dimension zero) is one dimensional. The generating element is the famous j -function,

$$j(q) = \frac{1}{1728} \left[\frac{1}{q} + 744 + 196844q + \dots \right] \quad (2.5)$$

As obvious from (2.5), $j(q)$ has a pole of order one at infinity. It is less obvious that it has a third order pole at $\tau = \rho$, ($n_\rho = 3$ in (2.4)). $j(q) - 1$ has a single pole at infinity and a double zero at $\tau = i$. Finally, $j'(q) \equiv q \frac{\partial}{\partial q} j(q)$ has a single pole at infinity, a single zero at $\tau = i$ and a double zero at $\tau = \rho$. These facts will be important in the sequel. A modular form is called entire if it is holomorphic, (no poles), in H/Γ . Any entire form can be represented as,

$$f_{m,n,l} = \frac{j^m}{j^n (j-1)^l}, \quad m, n, l \in \mathbf{Z} \quad (2.6)$$

with $2l \leq m, 3n \leq 2m, n+l \geq m$. Its dimension is $-2m$.

There is a subgroup of Γ that will be important in our discussion. This is the congruence subgroup at level two, $\Gamma(2)$. An element of $\Gamma(2)$ is of the form,

$$A \in \Gamma(2), \quad A = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{mod}(2) \quad (2.7)$$

$\Gamma(2)$ has index six in Γ and the coset representatives of $\Gamma/\Gamma(2)$ are generated by the elements, $1, T, S, TST, ST, (ST)^2$. Thus the fundamental domain, $H/\Gamma(2)$ is a six fold covering of H/Γ . Each point in H/Γ has six images in $H/\Gamma(2)$ except for $i\infty, i, \rho$. $H/\Gamma(2)$ has also genus zero and is diffeomorphic to the Riemann sphere. It will turn out that it will be more convenient to work on $H/\Gamma(2)$ rather than H/Γ in order to resolve the orbifold singularities. Let's introduce the $\Gamma(2)$ invariant coordinate x ,[‡]

$$x \equiv \left[\frac{\vartheta_2(\tau)}{\vartheta_3(\tau)} \right]^4 = 16q^{\frac{1}{2}} \prod_{n=1}^{\infty} \left[\frac{1+q^n}{1+q^{n-\frac{1}{2}}} \right]^8 \quad (2.8)$$

[‡]The coordinate in $H/\Gamma(2)$.

Under (2.8) the points $0, 1, \infty$ in $H/\Gamma(2)$ are mapped to $i\infty$ in H/Γ , the points $-1, 2, \frac{1}{2}$ in $H/\Gamma(2)$ are mapped to i in $H\Gamma$ and the points $e^{\pm \frac{i\pi}{3}}$ in $H/\Gamma(2)$ are mapped to ρ in $H\Gamma$.

The modular group acts on x as follows,

$$\begin{aligned} S : \tau &\rightarrow -\frac{1}{\tau} \quad , \quad x \rightarrow 1 - x \\ T : t &\rightarrow \tau + 1 \quad , \quad x \rightarrow \frac{x}{x-1} \end{aligned} \quad (2.9)$$

Let's introduce the following polynomials in x ,

$$Q_i(x) \equiv (x+1)(x-2)(x-\frac{1}{2}), \quad Q_\rho(x) \equiv x^2 - x + 1 \quad (2.10a)$$

$$\begin{aligned} Q_a(x) &\equiv (x-a)(x-\frac{1}{a})(x-1+a)(x-\frac{1}{1-a})(x+\frac{a}{1-a})(x+\frac{1-a}{a}) \\ &\equiv x^6 - 3x^5 + f(a)x^4 + [5 - 2f(a)]x^3 + f(a)x^2 - 3x + 1 \end{aligned} \quad (2.10b)$$

$$f(a) \equiv -\frac{a^6 - 3a^5 + 5a^3 - 3a + 1}{a^2(a-1)^2}, \quad a \neq 0, \pm 1, 2, \frac{1}{2}, e^{\pm \frac{i\pi}{3}} \quad (2.10c)$$

They have the property that their roots are permuted under the action of the modular group, (2.9). We can express $j(q)$ and its derivative in terms of x as follows,

$$\begin{aligned} j(q) &= \frac{4}{27} \frac{Q_\rho^3(x)}{x^2(x-1)^2} \\ j(q) - 1 &= \frac{4}{27} \frac{Q_i^2(x)}{x^2(x-1)^2} \\ j'(q) &= -\frac{4}{27} \vartheta_3^4(\tau) \frac{Q_i(x)Q_\rho^2(x)}{x^2(x-1)^2}, \quad q \frac{\partial}{\partial q} x = \frac{x}{2} \vartheta_4^4 \end{aligned} \quad (2.11)$$

In (2.11) one can read immediately the poles and zeros. In fact if a modular form, $f(\tau)$, has a pole (zero) of order $n \in Z_0^+$ (Z_0^-) at a point $\tau = \tau_0$ then,

$$f(x) = \frac{g(x)}{Q_a^n(x)}, \quad a = x(\tau_0) \quad (2.12)$$

where $g(x)$ is holomorphic in the neighborhood of $x = a$.

3. Constraints on the Characters of RCFT

In this section we are going to discuss physical constraints on the characters of RCFT.

By definition the partition function on the torus can be written as,

$$\mathbf{Z}(\tau, \bar{\tau}) = \sum_{i=0}^N f_i(\tau) \bar{g}_i(\bar{\tau}) \quad (3.1)$$

We call characters the holomorphic blocks, f_i, \bar{g}_i . From now on we will restrict ourselves in the study of the L-H characters, f_i to avoid repetition. The f_i furnish a finite dimensional projective representation of the modular group, Γ ,

$$f_i(A\tau) = M_{ij}(A)e^{i\phi(A)}f_j(\tau), \quad A \in \Gamma \quad (3.2)$$

where $M_{ij}(A)$ provide an N-dimensional representation of Γ and $e^{i\phi(A)}$ is a set of multipliers. They are determined by the multipliers of S and T . Because of $S^2 = (ST)^3 = 1$ we obtain $e^{2i\phi(S)} = e^{6i\phi(T)} = 1$ so that the multipliers are sixth-roots of unity. This is equivalent to the statement that $H(\Gamma, U(1)) = Z_6$.

The characters, f_i , are traces over the states of irreducible representations of the maximal chiral operator algebra of the theory,

$$f_i(q) = Tr_{R_i} \left[q^{L_0 - \frac{c}{24}} \right] \equiv q^{\Delta_i - \frac{c}{24}} \tilde{f}_i(q) \quad (3.3)$$

where Δ_i is the dimension of the highest weight state of the algebra. $\tilde{f}_i(0)$ is the multiplicity of the highest weight state. $\tilde{f}_i(q)$ has a Fourier expansion of the form,

$$\tilde{f}_i(q) = \sum_{n=0}^{\infty} a_n q^n \quad (3.4)$$

Consistency with the interpretation of $f_i(q)$ as a trace over orthonormal states in the Hilbert space of the theory implies that the Fourier coefficients in (3.4) must be integers. In particular since there exists a single unit operator in the theory we deduce that $\tilde{f}_0(0) = 1$. We must impose also that Δ_i be non-negative, since it is unphysical to have correlations growing with distance.[§]

There is a stronger constraint on the Fourier coefficients. Since conformal invariance is a symmetry of the theory the states must be arranged in irreducible representations of the Conformal Group. The partition function must be expressible in the form,

$$\mathbf{Z} = \sum_{ij} N_{ij} \chi_{\Delta_i} \bar{\chi}_{\bar{\Delta}_j} \quad (3.5)$$

where χ_i are the characters of the Conformal Group and the sum is necessarily infinite if $c \geq 1$. (3.5) imposes extra restrictions on the Fourier coefficients of the characters.

So far we said nothing about the analytic behavior of the characters in H/Γ . We will now show that the characters in RCFT must be holomorphic in H/Γ with possible poles only at $\tau = i\infty$.[¶] In particular they cannot have essential singularities, poles or

[§]There is no argument to our knowledge to prevent c from being negative, except unitarity but there are plenty of sensible critical systems which are non-unitary. In a unitary theory the Fourier coefficients of the characters must be non-negative. However positivity of the coefficients does not necessarily imply unitarity. An example of this can be found in section 6.

[¶]I would like to thank J. Cardy for suggesting the essential argument.

branch cuts inside H/Γ . To prove the previous claim we must study the convergence properties of $\tilde{f}_i(q) \equiv \sum_{n=0}^{\infty} a_n q^n$. The Verma module associated to the representation R_i of the chiral algebra of the theory is generated by the action of the lowering operators of the algebra, U , on the highest weight state. Let the number of local operators that generate U be $N \geq 0$. We will assume that the local operators in U have positive integral dimensions but we think that the results are almost certainly valid when the integrality condition is dropped. Let,

$$\prod_{n=1}^{\infty} (1 - q^n)^{-N} = \sum_{n=0}^{\infty} b_n q^n \quad (3.6)$$

Then $a_n \leq b_n, \forall n \in \mathbb{Z}_0^+$. Equality holds when there are no null vectors in the Verma modules. The asymptotic behaviour of b_n for n sufficiently large is,

$$b_n \sim n^{-a} \exp(2\pi \sqrt{\frac{N}{6}} \sqrt{n}), \quad a > 0 \quad (3.7)$$

Now we can prove the following: Let $f(q) = \sum_{n=0}^{\infty} a_n q^n$ with $|q| < 1$ and a_n behaves for large enough n as $P(n)n^{-a} \exp(b\sqrt{n})$, a, b being real and $P(n)$ is an arbitrary polynomial in n . By applying the n -th root convergence test it is obvious that the series converges absolutely. Since in $H/\Gamma, |q| \leq \exp(-\pi\sqrt{3}) < 1$ we can conclude that the characters do not have a singularity in the interior of H/Γ . Suppose now that a character has a branch cut somewhere. Then by differentiating enough times we can create a singularity at that point. But differentiation changes the asymptotic behavior of the Fourier coefficients by multiplication by a polynomial in n and by the previous argument this does not affect the convergence properties. Thus branch cuts cannot appear. As we will see in the sequel the above will prove to be crucially useful for our purposes.

4. Fuchsian Differential Equations on Γ and $\Gamma(2)$

It is a well known fact in Mathematics^{||} that if (f_i) is a set of n linearly independent meromorphic functions in a connected open subset of \mathbf{C} and D is a meromorphic vector field not identically zero, then there exist unique meromorphic functions, p_i , such that the space of solutions of,

$$D^n f + p_1 D^{n-1} f + p_2 D^{n-2} f + \dots + p_n f = 0 \quad (4.1)$$

is precisely $span(f_i)$.

A more abstract way of putting the above is that for every projectively flat, finite dimensional vector bundle over an algebraic curve, there is a differential equation the solutions of which form a basis for meromorphic sections of the bundle in question**

^{||}See for example [11].

**Some regularity conditions must be assumed.

In our case there are two convenient choices of coordinates. The first is the coordinate around a neighborhood of $i\infty$, q . The covariant derivative acting on forms of dimension $2k$ is $D_k \equiv q \frac{\partial}{\partial q} - \frac{k}{12} E_2(q)$ where $E_2(q) = -\frac{12i}{\pi} \frac{\partial}{\partial \tau} \log \eta(\tau)$, [12]. Then (4.1) can be written as,

$$D^n f + \sum_{i=1}^n g_i(q) D^i f = 0 \quad (4.2)$$

D^i is defined to be $D_{2i-2} D_{2i-4} \cdots D_0$. Since we want the solutions of (4.2) to furnish a representation of the modular group the coefficient functions g_i must be modular forms of dimension $2i$, [9]. For example the explicit form of the order two equation is,

$$\left(q \frac{\partial}{\partial q} - \frac{E_2(q)}{12} \right) q \frac{\partial}{\partial q} f + g_1(q) q \frac{\partial}{\partial q} f + g_2(q) f = 0 \quad (4.3)$$

where g_1, g_2 are modular forms of dimension $2(4)$. The modular forms g_i will generically have poles in H/Γ .

It will be convenient to use also the $\Gamma(2)$ invariant coordinate x defined in (2.8) in order to avoid complications with the ramification points of H/Γ . In this case too, it can be shown that if (f_i) generate a finite dimensional representation of the modular group then they are solutions of a differential equation of the form,

$$\frac{d^n}{dx^n} f + \sum_{i=1}^n p_i(x) \frac{d^i}{dx^i} f = 0 \quad (4.4)$$

The coefficients $p_i(x)$ must be rational functions of x . Equation (4.4) must be of the Fuchsian type, that is all its singularities must be regular. It will turn out that the representation (4.4) will be more convenient in studying general properties, but in actual computations (4.2) sometimes has a more practical value.

In order solve the constraints imposed by modular invariance on p_i we make a change of variables in (4.2) to map it to (4.4). For simplicity let's transform (4.3),

$$\begin{aligned} q \frac{\partial}{\partial q} &= \frac{1}{2} \vartheta_4^4(\tau) x \frac{d}{dx} \\ \frac{d^2}{dx^2} f + \left(\frac{2}{3} \frac{2x-1}{x(x-1)} + q_1(x) \right) \frac{d}{dx} f + q_2(x) f &= 0 \\ x q_1(x) &= 2 \frac{g_1(\tau)}{\vartheta_4^4(\tau)}, \quad x^2 q_2(x) = 4 \frac{g_2(\tau)}{\vartheta_4^4(\tau)} \end{aligned} \quad (4.5)$$

Since g_1, g_2 are modular forms, their transformation properties under Γ fix the transformation properties of q_1, q_2 under (2.9),

$$\begin{aligned} q_1\left(\frac{1}{x}\right) &= -x^2 q_1(x) & q_2\left(\frac{1}{x}\right) &= x^4 q_2(x) \\ q_1(1-x) &= -q_1(x) & q_2(1-x) &= q_2(x) \end{aligned} \quad (4.6)$$

One could understand the transformation laws in (4.6) as follows. The point $\tau = i\infty$ is a singular point of (4.3). This point corresponds to the three singular points, $0, 1, \infty$ of

(4.5). The exponents at these three singular points must be equal to $-\frac{c}{12}$, $-\frac{c}{12} + 2\Delta$ from the fact that the two solutions are characters. Thus (4.5) must be invariant when written in local coordinates around each of $0, 1, \infty$. This statement is equivalent to the statement that the solutions generate a two-dimensional representation of the modular group. This representation coincides with the monodromy representation of (4.5) around $0, 1, \infty$. The equation (4.5) must be a Fuchsian Differential Equation, (FDE), in order that the character not to have essential singularities. We remind the reader that for x_0 to be a regular singular point of (4.4), $(x-x_0)^i p_i(x)$ must be holomorphic in a neighborhood of x_0 . For example, $p_1(x)$ can have only first order poles and so on.

Consider now one of the coefficients, $g_k(q)$ in (4.2). As stated earlier, it is a modular form of dimension $2k$. Then by (2.4) the difference of the number of its zeros minus the number of its poles is $\frac{k}{6}$. As discussed above we also have the constraint that $p_k(x) \sim \frac{g_k(q)}{[x\vartheta_4^4(q)]^k}$ must have at most k -th order poles. Since g_k is a modular form, if $p_k(x)$ has a pole at $x = a$ then it must have poles of the same order at $x = 1 - a, \frac{1}{a}, \frac{1}{1-a}, \frac{a}{a-1}, \frac{a-1}{a}$ and its transformation properties under Γ imply that,

$$p_k\left(\frac{1}{x}\right) = (-1)^k x^{2k} p_k(x), \quad p_k(1-x) = (-1)^k p_k(x) \quad (4.7)$$

The constraints above fix the form of $p_k(x)$ to be,

$$p_k(x) \sim \frac{[Q_\rho^2(x)Q_i(x)]^k \prod_{i=1}^{k(N-1)} Q_{b_i}(x)}{[x(x-1) \prod_{i=1}^N Q_{a_i}(x)]^k} \quad (4.8)$$

where the polynomials $Q_a(x)$ have been defined in (2.10). The points a_i must be distinct. It should be also kept in mind that, $Q_{a \rightarrow i}(x) = Q_i^2(x)$, $Q_{a \rightarrow \rho}(x) = Q_\rho^3(x)$.

(4.8) is the most general form $p_k(x)$ can have, compatible with the requirements that (4.4) be a FDE and that its solutions generate an n -dimensional representation of the modular group. One can also express back in the q -coordinate using,

$$\begin{aligned} j(q) &\sim \frac{Q_\rho^3(x)}{x^2(x-1)^2}, \quad j(q) - 1 \sim \frac{Q_i^2(x)}{x^2(x-1)^2} \\ j'(q) &\sim \vartheta_3^4(q) \frac{Q_i(x)Q_\rho^2(x)}{x^2(x-1)^2}, \quad j(q) - a \sim \frac{Q_b(x)}{x^2(x-1)^2}, \quad x(q=a) = b \\ g_k(q) &= [x\vartheta_4^4(q)]^k p_k(x) \end{aligned} \quad (4.9)$$

to obtain the most general allowed form of g_k ,

$$g_k(q) \sim \frac{(j'(q))^k \prod_{i=1}^{k(N-1)} [j(q) - b_i]}{\prod_{i=1}^N [j(q) - a_i]^k}, \quad a_i \text{ distinct} \quad (4.10)$$

The explicit form of the general n -th order equation (4.4) is given in appendix A.

We will now come to an important point, namely the condition of regularity of solutions in $H/\Gamma(2) \simeq \hat{\mathbf{C}}$. We have proven in section 3 that the characters of RCFT's

cannot have poles branch cuts or other singularities in $H/\Gamma(2) - \{0, 1, \infty\}$. This will add a crucial restriction on the FDE. Consider x_0 to be a singular point of (4.4) other than $0, 1, \infty$. The behaviour of a set of solutions, $f_i(x)$ around this point is of the form: $f_i(x) = Q_{x_0}^{N_i}(x)h_i(x)$ with $h_i(x)$ holomorphic in the neighborhood of x_0 .^{††} The absence of a singularity or a branch point there implies that the numbers n_i must be non-negative integers. This puts a restriction on the complex coefficients that $p_i(x)$ depend upon.

To this point we did not discuss the possibility that solutions of the FDE contain logarithms. Such a situation is unacceptable from the point of view of RCFT. Let's again focus on a specific singular point x_0 of the FDE and let m_i be the indices of that singular point. The solutions will contain logarithmic terms around x_0 if two of the indices differ by an integer^{‡‡}. If at least two of the indices are equal then there is no way to get rid of the logarithmic terms. When the singular point in question is one of $0, 1, \infty$, then the indices are nothing else than $-\frac{c}{12} + 2\Delta_i$. The case of having two of them equal can be excluded by definition since this would imply that the characters should satisfy an equation of order $n-1$. But if critical dimensions differ by integers then one should impose that the solutions be free of logarithms. For the rest of the singular points, as we argued above, the indices are positive integers. Thus the problem needs to be always confronted there. In particular it is immediate that the integers should be pairwise distinct. We will come again to this point in the next section.

5. The Riemann-Hilbert Problem, Apparent Singularities and Isomonodromic Deformations

Let $\hat{\mathbf{C}}$ be the Riemann sphere and $S \equiv \{x_1, x_2, \dots, x_m\}$ be a collection of mutually distinct points. $\hat{\mathbf{C}} - S$ is the m -punctured sphere with punctures in S . Let $\pi_1(\hat{\mathbf{C}} - S)$ be the fundamental group of $\hat{\mathbf{C}} - S$. $\pi_1(\hat{\mathbf{C}} - S)$ is a free group generated by $m-1$ elements, which we can take to be the loops around x_1, x_2, \dots, x_{m-1} .

The Riemann-Hilbert Problem: Let $\rho : \pi_1(\hat{\mathbf{C}}) \rightarrow GL(n, \mathbf{C})$ be a homeomorphism. Find an n -th order FDE having ρ as its monodromy representation, (we assume that ρ is irreducible over \mathbf{C}^n).

To motivate the solution to the previous problem, let's do some counting. Assume without loss of generality that $x_m = \infty$. An n -th order FDE with singularities on S has the following form,

$$f^{(n)} + \frac{P_{m-2}(x)}{g(x)} f^{(n-1)} + \frac{P_{2(m-2)}(x)}{g(x)^2} f^{(n-2)} + \dots + \frac{P_{n(m-2)}(x)}{g(x)^n} f = 0 \quad (5.1)$$

^{††}The numbers N_i are known as the indices of the FDE at x_0 .

^{‡‡}See any standard textbook on linear differential equations.

where,

$$g(x) = \prod_{i=1}^{m-1} (x - x_i) \quad (5.2)$$

and $P_i(x)$ are polynomials of degree i . A simple counting of the coefficients in (5.1) shows that there are $N_1 = n + \frac{n(n+1)(m-2)}{2}$ free (complex) parameters. Since $\rho : \pi_1(\hat{\mathbf{C}}) \rightarrow GL(n, \mathbf{C})$, ρ depends on $N_2 = n^2(m-1) - (n^2 - 1)$ parameters. It should be remembered that the image of ρ in $GL(n, \mathbf{C})$ matters up to conjugacy. The difference is $N_2 - N_1 = \frac{(n-1)(n(m-2)-2)}{2}$. We see that generically $N_2 - N_1 > 0$ so that if one expects to have a solution to the Riemann-Hilbert (R-H) problem, more parameters must be introduced in (5.1) somehow. This is in fact the role of apparent singularities.

An apparent singularity is defined to be a singularity of (5.1) around which the (local) monodromy is trivial. It is obvious that the following four statements are equivalent.

- (1). The point x_0 is an apparent singularity of (5.1).
- (2). The point x_0 is a singularity of (5.1) and there exist n linearly independent holomorphic solutions in a neighborhood of x_0 .
- (3). The point x_0 is a non-logarithmic singularity of (5.1) with positive integral indices.
- (4). There exist n linearly independent solutions around x_0 whose Wronskian vanishes at x_0 .

Let's see why the presence of apparent singularities increases the number of free parameters in (5.1). Suppose that (5.1) has an extra apparent singularity at $x = \zeta$. Then the number of free parameters, N_1 , increases by $\frac{n(n+1)}{2}$. But the requirement that $x = \zeta$ be an apparent singularity imposes $\frac{n(n+1)}{2}$ conditions on the coefficients of (5.1), n of them come from the condition that the n indices around $x = \zeta$ be positive integers and $\frac{n(n-1)}{2}$ from imposing that the n solutions around $x = \zeta$ be free from logarithms*. Thus we gained nothing so far. But on the other hand we can vary the position of the apparent singularity around and this gives us an extra free parameter. Consequently each apparent singularity of (5.1) increases N_1 by one. Whether ζ moves or not, it does not change N_2 since by construction the local monodromy around ζ is trivial. After this motivation we can state the main theorem, [13].

Theorem: Let $\hat{\mathbf{C}}$ and S be as above and $\rho : \pi_1(\hat{\mathbf{C}} - S) \rightarrow GL(n, \mathbf{C})$ a homeomorphism. Assume that ρ is irreducible and assume further that there exists an index i such that the monodromy matrix around $x_i \in S$ has at least one Jordan block of size one. Then there exists a FDE on $\hat{\mathbf{C}}$ with regular singularities on $S \cup S'$ whose monodromy representation is ρ where S' consists of at most $N_2 - N_1$ apparent singularities.

We recognize now, coming back to the discussion of the previous section, that the

*See any standard textbook on linear differential equations.

number of apparent singularities of (4.4) is limited by the previous theorem. First as it was already mentioned, $\rho : \pi_1(\hat{\mathbf{C}} - \{0, 1, \infty\}) \rightarrow GL(n, \mathbf{C})$ is a representation of the modular group, Γ . All other singularities of (4.4) are apparent singularities and their number must be at most $\frac{1}{2}(n-1)(n-2)$. For $n = 2$ there can't be any apparent singularities. For $n = 3$ there can be at most one apparent singularity. But from the modular invariance of the coefficient functions we know that singularities come in groups of two ($\tau = e^{\frac{2\pi i}{3}}$), three ($\tau = i$) or six (any other τ). Thus for $n = 3$ there can't be any apparent singularities. For $n = 4$ we can have at most three of them so that they can be only at the ramification points, and so on. If one insists on unitarity then apparent singularities of the FDE, (zeros of the characters) at τ being on the imaginary axis should be avoided [†]

Once the answer to the R-H problem is settled, the question of isomonodromic deformations becomes an issue. The problem is: given two FDE's, when they generate the same monodromy representation. The answer to this question is known in the case where there are continuous families of isomonodromic deformations, [14]. The solution is given implicitly as a Hamiltonian system involving non-linear partial differential equations. Fortunately in our case this is not an issue since the only parameters that one can vary are the positions of the true singularities, but these are fixed in our case at $x = 0, 1, \infty$.

The only procedure, in our case, that does not change the monodromy representation is multiplying each character independently by a modular invariant function which is rational in x . Such functions are rational functions of $\frac{Q_\alpha(x)}{[x(x-1)]^2}$, (which is nothing else than $j(q)$ + constant). One should be careful not to introduce singularities that way. In RCFT we are interested in projective representations of the modular group. We have already pointed out that the multiplier system for Γ is a subgroup of Z_6 . This is generated when we allow multiplication with products of powers of $\frac{Q_\rho(x)}{[x(x-1)]^{\frac{2}{3}}}$, $\frac{Q_i(x)}{x(x-1)}$, [8]. In the q -coordinate it is equivalent to multiplying the characters by $j(q)^{\frac{N}{3}} [j(q) - 1]^{\frac{M}{2}}$, with M, N integers.

The above concludes our general analysis and we are ready to attack a concrete problem. In the next section we are going to classify all RCFT's with two characters.

5. Classification of RCFT's with two Characters

In the previous section we showed that for $n=2$ there can be no apparent singularities.

[†]If $Re\tau = 0$ then q^n is strictly positive $\forall n \in Z_0^+$. thus in order that $f_i(q)$ have a zero on the imaginary axis it is necessary that some of its Fourier coefficients to be negative. This is in contradiction with unitarity.

Thus the FDE appropriate to this case is,[‡]

$$f'' + \frac{2}{3} \frac{2x-1}{x(x-1)} f' - \mu \frac{x^2-x+1}{x^2(x-1)^2} f = 0 \quad (6.1)$$

which can be transformed to the standard Hypergeometric Equation, (HE). Set $f(x) = [x(x-1)]^n g(x)$ with n satisfying $n^2 - \frac{n}{3} - \mu = 0$. Then (6.1) transforms into,

$$x(1-x)g'' + \left(\frac{2}{3} + 2n\right)(1-2x)g' - n(3n+1)g = 0 \quad (6.2)$$

which is the standard HE with $a = 3n$, $b = n + \frac{1}{3}$, $c = 2n + \frac{2}{3}$. The two independent solutions of (6.1) are,

$$\begin{aligned} f_0(x) &= 2^{-4n} [x(1-x)]^n F\left(3n, n + \frac{1}{3}, 2n + \frac{2}{3}, x\right) \\ f_1(x) &= 2^{4n-\frac{4}{3}} x^{-n+\frac{1}{3}} (1-x)^n F\left(-n + \frac{2}{3}, n + \frac{1}{3}, -2n + \frac{4}{3}, x\right) \end{aligned} \quad (6.3)$$

where we (arbitrarily) normalized both solutions so that $f_i(q) = q^a [1 + O(q)]$. Recalling the definition of x in terms of q , (2.8), we can identify $n = -\frac{c}{12}$ and $\Delta = \frac{c+2}{12}$.

A major constraint turns out to be the integrality condition on the Fourier coefficients of $f_0(q)$. In [9] it was shown that integrality at the first level selects a finite number of values for c .[§] In table 1 we present the values of c for which the Fourier coefficients of the identity character are integers. In order to choose the identity character unambiguously one has to restrict to $\Delta > 0$. A priori we cannot say anything about the Fourier coefficients of the non-trivial character $f_1(q)$ since we do not know the multiplicity of its ground state, although in the sequel we will provide with a procedure that determines it.

There are a few comments in order concerning the contents of the table. The first ten models have positive integral coefficients in f_0 . Of course this by itself does not guarantee unitarity since the model with $c = \frac{2}{5}$ is known not to be unitary. The other nine models are unitary since they can be identified with known models, [9]. The models with $c = 12, 13, 14, 16, 18, 19, \frac{62}{5}, \frac{74}{5}, \frac{86}{5}, \frac{98}{5}, \frac{194}{5}$ have all coefficients being negative integers. It is thus impossible to construct a character with positive integral coefficients by multiplication by a $Q_a(x)$ polynomial. The rest of the models have their coefficients positive integral except the first and in some the second. In this case it is easy to arrange that by multiplication with $Q_a^N(x)$ they become positive integers. Thus the complete list of models, (unitary or not), having two characters have as central charges the ones depicted in table 1 modulo(8,12,24).[¶]

We now come to the important issue of determining the dimensionality of the ground state of the non-trivial representation, that is the normalization of $f_1(q)$. It turns out

[‡]See also [9] where (6.1) was also discussed.

[§]The authors of reference [9] considered only the solutions that give positive Fourier coefficients.

[¶]In fact the theories with $c = \frac{38}{5}, \frac{98}{5}$ do not correspond to RCFT's since they fail to satisfy the Verlinde algebra, (the dimensionalities of bundles on the 3-punctured sphere fail to be non-negative integers, [9]).

that by studying the modular transformation matrix, S_{ij} , we can fix this normalization unambiguously. Let's recall the transformation formula for the hypergeometric functions,

$$F(a, b, c, 1-x) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a, b, a+b+1-c, x) + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} x^{c-a-b} F(c-a, c-b, c+1-a-b, x) \quad (6.4)$$

From (6.3) and (6.4) we can find the expression for the modular transformation matrix S_{ij} defined by $\tau \rightarrow -\frac{1}{\tau}$, $f_i \rightarrow S_{ij} f_j$,

$$S_{11} = -S_{22} = \frac{\Gamma(\frac{4-c}{6})\Gamma(\frac{c+2}{6})}{\Gamma(\frac{c+8}{12})\Gamma(\frac{4-c}{12})} \quad (6.5a)$$

$$S_{12} = 2^{\frac{2c+4}{3}} \frac{\Gamma(\frac{4-c}{6})\Gamma(-\frac{c+2}{6})}{\Gamma(-\frac{c}{4})\Gamma(\frac{4-c}{12})} \quad (6.5b)$$

$$S_{21} = 2^{-\frac{2c+4}{3}} \frac{\Gamma(\frac{c+8}{6})\Gamma(\frac{c+2}{6})}{\Gamma(\frac{c+2}{12})\Gamma(\frac{c+4}{4})} \quad (6.5c)$$

There are two facts apparent in (6.5),

$$\det(S) = -1, \quad \text{Tr}(S) = 0 \quad (6.6)$$

Let the partition function be,

$$Z = f_i M_{ij} \bar{f}_j \quad (6.7)$$

Since none of the theories in table 1 has $\Delta = \text{integer}$, invariance under T implies $M_{12} = M_{21} = 0$. The identity character, $f_0(q)$ is normalized such that, $f_0(q) = q^{-\frac{c}{24}} [1 + O(q)]$. Consequently, $M_{11} = 1$. Since $f_1(q)$ is also normalized as $f_1(q) = q^{-\frac{c}{24} + \Delta} [1 + O(q)]$ then $M_{22} = rN^2$ with N a positive integer being the dimensionality of the ground state of the non-trivial representation. Invariance under S implies that $rN^2 = \frac{S_{12}}{S_{21}}$ where r is the number of times the non-trivial character appears. From this and (6.5) we obtain,

$$rN^2 = -\frac{\Gamma(-\frac{c+2}{12})\Gamma(-\frac{c+2}{6})\Gamma(-\frac{c+2}{4})\Gamma(\frac{c+2}{2})}{\Gamma(\frac{c+2}{12})\Gamma(\frac{c+2}{6})\Gamma(\frac{c+2}{4})\Gamma(-\frac{c+2}{2})} \quad (6.8)$$

Direct evaluation of (6.8) gives the dimensionality of the ground state of the non-trivial representation tabulated in table 1 and thus provides the correct normalization for f_1 . For the known cases this agrees with the standard answer.^{||} In all cases the characters reduce to algebraic functions of x since they are invariant under some $\Gamma(N)$. We present such expressions in Appendix B.

To conclude this section, we gave a complete classification of the RCFT's with two characters and gave the explicit expressions for their characters. It is interesting to analyze theories with more than two characters.

^{||}In the cases $c = 8, 12, 20$, $N_0 = 0$. This means that these theories contain a single character, namely f_0 . This is well known to be true in the E_8 case.

7. FDE's for other Correlation Functions

The four-point functions on the sphere have singularities only at $x = 0, 1, \infty$, where x is the standard anharmonic ratio of the positions of the four operators** In this case our analysis carries out unchanged. The monodromy group is generated by the same transformations as in (2.9). On here does not have the restriction of integrality of coefficients but there are some other restrictions on the coefficients, [15].

The previous analysis carries out unchanged also to one-point functions on the torus. Due to translation invariance on the torus one-point functions are independent of the position of the operator and depend only on τ . They transform also as forms of dimension Δ , where Δ is the dimension of the operator in question. Let $f_i(\tau)$ be such a collection of one-point functions. Then if we define $g_i(\tau) \equiv \frac{f_i(\tau)}{\eta^{2\Delta}(\tau)}$, where $\eta(\tau)$ is the Dedekind η -function, then the g_i generate a representation of Γ ,

$$g_i(A\tau) = A_{ij}g_j(\tau), \quad A \in \Gamma \quad (7.1)$$

where A_{ij} is a τ -independent matrix. Thus one can write FDE's for the g_i which are the same as the ones for the characters.

The most interesting case is that of the two-point functions on the torus. This case was also discussed in [9]. The torus can be mapped onto the four-punctured sphere by the Weierstrass function, $\wp(z)$. Thus, if we introduce the coordinate $\xi = \wp(z)$ then the 2-point functions must satisfy a standard FDE on the sphere with singular points at $\xi = 0, 1, \infty, x$ where x is the same as in (2.8). Since now we have a variable singular point the FDE's for the two-point functions have more diversity than those for the characters. The two-point functions on the torus are a vital ingredient in CFT since they contain all the information that is required to specify the CFT, (at least in the rational case, see [4]).

We are going to present a general analysis of FDE's for the two-point functions in a separate publication.

8. Conclusions and Prospects

In this paper we studied an analytic aspect of RCFT's. We used the fact that the characters on the torus have to satisfy a FDE to provide a classification of such equations. We discussed the restrictions, coming from physical arguments, that their solutions must satisfy.

**Differential equations for the four-point functions have been recently discussed in [15].

For the special case of theories with two characters we gave a complete classification and explicit expressions for the characters. We briefly discussed also other applications of the FDE method and the similarities to the case at hand.

It seems imperative to extend this analysis to 2-point functions on the torus, due to their importance, since they are encompassing all data needed to define a RCFT unambiguously. It also seems likely that the specification of a complete consisted set of data à la Moore-Seiberg, uniquely determines the Linear Differential Equations satisfied by the correlation functions and thus provides a more or less complete solution of the theory. Such questions need to be addressed.

Note Added, I

It seems that characters of WZW-models have no zeros in the fundamental region. This is true for $SU(2)$, $SU(3)$ and the minimal models with $c < 1$. It is probably true for any WZW-model. In this case one can then show that the characters of any G/H model have no zeros in the fundamental region. Thus this will be a test of a RCFT being a G/H model or not in the sense that if the characters have zeros in moduli space then the theory is not a G/H model.

Note Added, II

After the completion of this work we received the last of references [9] where, among other things, the authors classified all RCFT's with three characters and no zeros in moduli space. In view of our results this classification is complete.

Appendix A

In this appendix we are going to derive the form of the most general n -th order equation in x satisfied by a collection of n characters. Let ∇_k be the covariant derivative on forms of dimension k , [12]. Its form is,

$$\nabla_k = \frac{\partial}{\partial \tau} - \frac{i\pi k}{6} E_2(\tau) = \frac{\partial}{\partial \tau} - 2k \frac{\partial}{\partial \tau} \log \eta(\tau) \quad (\text{A.1})$$

Then the n -th order equation can be written as,

$$\partial^n + \sum_{k=1}^n g_k(\tau) \partial^{n-k} f = 0 \quad (\text{A.2})$$

where,

$$\partial^k \equiv \nabla_{2(k-1)} \nabla_{2(k-2)} \cdots \nabla_2 \nabla_0 \quad (\text{A.3})$$

and $g_k(\tau)$ are modular forms of dimension $2k$.

Let's introduce $x = \left[\frac{\vartheta_2(\tau)}{\vartheta_3(\tau)} \right]^4$ and define,

$$D_k \equiv x \frac{\partial}{\partial x} + k \frac{x+1}{3(x-1)} \quad (\text{A.4})$$

Then,

$$\partial^k = \left[i\pi \vartheta_4^4(\tau) \right]^k \sum_{i=0}^k A_i^k(x) \left(x \frac{\partial}{\partial x} \right)^i \quad (\text{A.5})$$

The coefficient functions $A_k^N(x)$ satisfy the recursion formula,

$$A_k^N(x) = A_{k-1}^{N-1}(x) + D_{N-1} A_k^{N-1}(x) \quad (\text{A.6})$$

From (A.6) and the condition $A_0^N = 0$ we can evaluate them to obtain,

$$A_k^N(x) = \sum_{i_1 > i_2 > \cdots > i_{N-k}=1}^{N-1} D_{i_1} D_{i_2} \cdots D_{i_{N-k}} 1 \quad (\text{A.7})$$

where the order of the factors in the product is from left to right and all the products of the differential operators act on the constant function 1.

Finally the functions $p_k(x) = g_k(\tau) [\vartheta_4(\tau)]^{-k}$ are $\Gamma(2)$ invariant and thus rational functions of x . For example the third order equation is,

$$\begin{aligned} f''' + \left[\frac{4x-2}{x(x-1)} + p_1(x) \right] f'' + \left[\frac{20x^2 - 20x + 2}{9x^2(x-1)^2} + \frac{2}{3} \frac{2x-1}{x(x-1)} p_1(x) + p_2(x) \right] f' + \\ + p_3(x) f = 0 \end{aligned} \quad (\text{A.8})$$

where the functions $p_k(x)$ must transform as indicated in (4.7).

Appendix B

In this appendix we present formulae for the characters for some of the theories in Table 1 in the case that they reduce to algebraic functions of x . In order to obtain them we had to use some duplication formulae for the hypergeometric functions, [16]. We define:

$$x_{\pm} = -(\sqrt{x} \pm \sqrt{x-1})^2 \quad (B.1)$$

Then,

$$c = 1 : f_0(x) = 2^{-\frac{2}{3}} [x(1-x)]^{-\frac{1}{12}} [\sqrt{1+\sqrt{x}} + \sqrt{1-\sqrt{x}}] \quad (B.2a)$$

$$f_1(x) = 2^{-\frac{2}{3}} [x(1-x)]^{-\frac{1}{12}} [\sqrt{\sqrt{x-1} + \sqrt{x}} - \sqrt{\sqrt{x-1} - \sqrt{x}}] \quad (B.2b)$$

$$c = 4 : f_0(x) = 2^{\frac{4}{3}} [x(1-x)]^{-\frac{1}{3}} \quad (B.3a)$$

$$F_1(x) = 2^{\frac{1}{3}} [x(1-x)]^{-\frac{1}{3}} x \quad (B.3b)$$

$$c = 7 : f_0(x) = 2^{-\frac{5}{3}} [x(1-x)]^{-\frac{7}{12}} [x_+^{-\frac{7}{4}} (1 + 7x_+^2) + (+ \rightarrow -)] \quad (B.4a)$$

$$c = 8 : f_0(x) = 2^{\frac{8}{3}} [x(1-x)]^{-\frac{2}{3}} Q_{\rho}(x) \quad (B.5)$$

$$c = 12 : f_0(x) = 2^4 [x(1-x)]^{-1} Q_i(x) \quad (B.6)$$

$$c = 13 : f_0(x) = -2^{-\frac{8}{3}} [x(1-x)]^{-\frac{13}{12}} [x_+^{-\frac{13}{4}} (1 - 26x_+^2 - 39x_+^4) + (+ \rightarrow -)] \quad (B.7)$$

$$c = 16 : f_0(x) = 2^{\frac{16}{3}} [x(1-x)]^{-\frac{4}{3}} (1 - 2x) \quad (B.8a)$$

$$f_1(x) = 2^{\frac{16}{3}} [x(1-x)]^{-\frac{4}{3}} x^3 (1 - \frac{x}{2}) \quad (B.8b)$$

$$c = 19 : f_0(x) = 2^{-\frac{11}{3}} [x(1-x)]^{-19/12} [x_+^{-\frac{19}{4}} (1 - 19x_+^2 - 285x_+^4 - 209x_+^6) + (+ \rightarrow -)] \quad (B.9)$$

$$c = 20 : f_0(x) = 2^{\frac{20}{3}} [x(1-x)]^{-\frac{5}{3}} Q_i(x) Q_{\rho}(x) \quad (B.10)$$

$$c = 25 : f_0(x) = 2^{-\frac{11}{3}} [x(1-x)]^{-\frac{25}{12}} [x_+^{-\frac{25}{12}} (1 - 20x_+^2 + 630x_+^4 + 2380x_+^6 + 1105x_+^8) + (+ \rightarrow -)] \quad (B.11)$$

$$c = 28 : f_0(x) = 2^{\frac{28}{3}} [x(1-x)]^{-\frac{7}{3}} \left(1 - \frac{7x}{2} + \frac{7x^2}{2}\right) \quad (B.12a)$$

$$f_1(x) = 7 \cdot 2^{\frac{22}{3}} [x(1-x)]^{-\frac{7}{3}} x^5 \left(1 - x + \frac{2x^2}{7}\right) \quad (B.12b)$$

$$c = 40 : f_0(x) = 2^{\frac{40}{3}} [x(1-x)]^{-\frac{10}{3}} (1-2x)(1-3x+3x^2) \quad (B.13a)$$

$$f_1(x) = 3 \cdot 2^{\frac{40}{3}} [x(1-x)]^{-\frac{10}{3}} x^7 \left(1 - 2x + \frac{10x^2}{7} - \frac{5x^3}{14}\right) \quad (B.13b)$$

For $c=8,12,20$ there is only one character, namely $f_0(x)$. For the other cases where $f_1(x)$ is omitted, it can be calculated from the transformation property under $\tau \rightarrow -\frac{1}{\tau}$, see (6.6).

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Table Caption

In table 1, c is the central charge, Δ and N_0 are the dimension and multiplicity of the ground state of the non-trivial representation, and N_1 and N_2 are the number of (1,0) and (2,0) operators in the identity module. Finally μ is the value of the coefficient in (6.1).