CALT-68-1510
DOE RESEARCH AND
DEVELOPMENT REPORT

Some Proofs on the Classification of Rational Conformal Field Theories with $c = 1$ ⋆

Elias B. Kiritsis †

California Institute of Technology, Pasadena, CA 91125

Abstract

Some detailed proofs are presented which complete the classification of rational conformal field theories with $c = 1$.

October 1988

⋆ Work supported in part by the U.S. Department of Energy under Contract DEAC-03-81-ER40050.

† Present address: Physics Dept. University of California, Berkeley, CA 94720.

In ref. [1] the author gave a proof asserting that the list of rational conformal field theories, given in [2] was complete. The first ingredient of the proof was showing that all possible modular invariants are toroidal partition functions. The second ingredient was in constructing all possible linear combinations of invariants that are partition functions in Conformal Field Theory (CFT). Concerning the second part, a series of propositions were put forth in [1] without proof. The idea of this line of reasoning is due to ref. [3]. We present the proofs here for completeness and to make the arguments in [1] and [3] more transparent. We will employ the notation of ref. [1].

The toroidal partition functions are defined as:

$$\mathbf{Z}(R) \quad = \quad \frac{1}{\eta\bar{\eta}} \sum_{m,n\epsilon Z} q^{\frac{1}{2}P_L^2} \bar{q}^{\frac{1}{2}P_R^2} \tag{1}$$

with $P_L = \frac{m}{R} + \frac{nR}{2}$, $P_R = \frac{m}{R} - \frac{nR}{2}$, $R > 0$ is the radius.

The partition function (1) is modular invariant, and is also invariant under the duality transformation $R \to \frac{2}{R}$. Thus we will restrict ourselves for the rest of this paper to $R \geq 2$. Define also $\mathbf{Z}_N \equiv \mathbf{Z}(R = N\sqrt{2})$. In order to have rational critical indices, $R = \sqrt{q}$, $q \in Q$. In [1] it was shown that all possible modular invariants at $c = 1$ are linear combinations of toroidal partition functions. Thus a general partition function can be written as a finite linear combination: $\mathbf{Z} = \sum_{i=1}^{N} c_i \mathbf{Z}(R_i)$, where c_i are some real coefficients. $\mathbf{Z}$ is automatically modular invariant. There are two extra constraints on $\mathbf{Z}$ though. First there should be a unique $(0,0)$ operator and second all the multiplicities have to be non-negative integers‡

We will now try to find all such linear combinations satisfying the above requirements.

We will introduce the Virasoro characters at $c = 1$ since they will be important in our proof. The characters are:

$$\begin{aligned} &\Delta \neq \frac{n^2}{4} \quad , \quad n \in Z \quad , \quad \chi_\Delta = \frac{q^\Delta}{\eta} \\ &\Delta = \frac{n^2}{4}, \quad , \quad n \in Z \quad , \quad \chi_{\frac{n^2}{4}} = \frac{q^{\frac{n^2}{4}}(1 - q^{n+1})}{\eta} \end{aligned} \tag{2}$$

‡ There is an extra constraint but we will come back to it at the end.

and if $\Delta = \frac{n^2}{4}$, $n \in Z$ then,

$$\frac{q^{\frac{n^2}{4}}}{\eta} = \sum_{k=0}^{\infty} \chi_{\frac{(n+2k)^2}{4}}$$

As usual $q = e^{2\pi i\tau}$ and $\eta(\tau)$ is the Dedekind η-function. Then we can express the toroidal partition functions (1) as sesquilinear combinations of Virasoro characters. The relevant formulas are as follows:

$$\mathbf{Z}_1 = \sum_{\substack{m=0\\n=0}}^{\infty} (m+1)(n+1)\chi_{\frac{m^2}{4}}\bar{\chi}_{\frac{n^2}{4}} \tag{3a}$$

$$\begin{aligned}\mathbf{Z}_N \quad &= \sum_{\lambda=0}^{2N-1} \Bigg\{ [\sum_{\substack{m=0\\k=0}}^{\infty} \chi_{\frac{(2Nm+2k+\lambda)^2}{4}} + \sum_{\substack{m=1\\k=0}}^{\infty} \chi_{\frac{(2Nm+2k-\lambda)^2}{4}}] \times \\ &\times [\sum_{\substack{m=0\\ \ell=0}}^{\infty} \bar{\chi}_{\frac{(2Nn+2\ell+\lambda)^2}{4}} + \sum_{\substack{n=1\\ \ell=0}}^{\infty} \bar{\chi}_{\frac{(2Nn+2\ell-\lambda)^2}{4}}] \Bigg\} \\ &+ \sum_{\substack{m,n\epsilon Z\\ m\neq 0 \bmod(N)}} \chi_{\frac{(m+nN^2)^2}{4N^2}} \bar{\chi}_{\frac{(m-nN^2)^2}{4N^2}} \end{aligned} \tag{3b}$$

Let $p, q \in Z$, $(p,q) = 1$.

$$\begin{aligned}\mathbf{Z}(R = \sqrt{2}\,\frac{p}{q}) = \sum_{\alpha=1}^{q-1}\sum_{\beta=1}^{p-1} \Bigg\{ &[\sum_{\substack{n=0\\k=0}}^{\infty} \chi_{\frac{(2mpq+\beta q+\alpha p+2k)^2}{4}} + \sum_{\substack{m=1\\k=0}}^{\infty} \chi_{\frac{(2mpq-\beta q-\alpha p+2k)^2}{4}}] \\ &\times [\sum_{\substack{n=0\\ \ell=0}}^{\infty} \bar{\chi}_{\frac{(2npq+\beta q-\alpha p+2\ell)^2}{4}} + \sum_{\substack{n=1\\ \ell=0}}^{\infty} \bar{\chi}_{\frac{(2npq-\beta q+\alpha p+2\ell)^2}{4}}] \\ &\times [\sum_{\substack{m=0\\k=0}}^{\infty} \chi_{\frac{(2mpq+\beta q+\alpha p+2k+pq)^2}{4}} + \sum_{\substack{m=1\\k=0}}^{\infty} \chi_{\frac{(2mpq-\beta q-\alpha p-pq+2k)^2}{4}}] \\ &\times [\sum_{\substack{n=0\\ ell=0}}^{\infty} \bar{\chi}_{\frac{(2npq+\beta q-\alpha p+2\ell)^2}{4}} + \bar{\chi}_{\frac{(2npq-\beta q+\alpha p-pq+2\ell)^2}{4}}] \Bigg\}\end{aligned}$$

$$+\sum_{\substack{m\neq 0 \bmod(p)\\ n\neq 0 \bmod(q)}} \chi_{\frac{(mq^2+np^2)^2}{4p^2q^2}} \quad \bar{\chi}_{\frac{(mq^2-np^2)^2}{4p^2q^2}} \tag{3c}$$

If $R \notin \sqrt{2}\,Q$ then:

$$\mathbf{Z}(R) \quad = \quad \sum_{P_L,P_R} \chi_{\frac{P_L^2}{2}} \quad \bar{\chi}_{\frac{P_R^2}{2}} \tag{3d}$$

The characters of a $\mathcal{U}(1)$ algebra are always given by

$$ch_Q = \frac{q^{\frac{Q^2}{2}}}{\eta} \tag{4}$$

where Q is the $\mathcal{U}(1)$ charge. They are the same as the Virasoro characters if the representation is not degenerate (it is degenerate if $\Delta = \frac{n^2}{4}, n \in Z$). Thus a non-degenerate Virasoro representation contains a single $\mathcal{U}(1)$ representation, whereas a $\mathcal{U}(1)$ representation with $\frac{Q^2}{2} = \frac{n^2}{4}, n \in Z$ decomposes into an infinite number of Virasoro representations. If a CFT has a $(1,0)$ operator that generates a $\mathcal{U}(1)$ affine algebra then the spectrum must be arranged in representations of this $\mathcal{U}(1)$ and the partition function must be a sesquilinear form of $\mathcal{U}(1)$ characters. If no $(1,0)$ operator is present, then the partition function must be a sesquilinear form of Virasoro characters.

We now proceed to the main line of proof.

Lemma 1: Let $R > \sqrt{2}$. Then the Virasoro representation $(\Delta, \bar{\Delta}) = (\frac{1}{2}(\frac{1}{R}+\frac{R}{2})^2\,,\,\frac{1}{2}(\frac{1}{R}-\frac{R}{2})^2)$ is not degenerate and appears exactly twice in $\mathbf{Z}(R)$.

Proof: Let $R = \sqrt{\frac{p}{q}}\,,\, p,q \in Z^+\,,\,(p,q) = 1$. Then $\Delta = \frac{1}{8pq}(2q+p)^2$.

In order for this to be degenerate, there must exist $N \in Z^+$ such that $\frac{1}{8pq}(2q+p)^2 = \frac{N^2}{4}$ or $(2q+p)^2 = 2pqN^2$.

(i) Suppose that $p = 2p', p' \in Z^+$, then $(q+p')^2 = p'qN^2$. This can be true if $p' = \kappa^2, \kappa \in Z^+, q = \lambda^2, \lambda \in Z^+, (\kappa, \lambda) = 1$. Then $\kappa^2 + \lambda^2 = \kappa\lambda N$. Let $\lambda > 1$ then $\lambda \mid \kappa^2$ wrong by assumption. If $\lambda = 1$ then $\kappa \mid 1$, so $\kappa = 1$ wrong since $R > \sqrt{2}$.

(ii) Suppose that 2 does not divide p, then $(2q+p)^2 = 2pqN^2$ and a solution exists if $p=\kappa^2, q=2\lambda^2, \kappa,\lambda \in Z^+, (\kappa,\lambda)=1$. Then $4\lambda^2+\kappa^2 = 2\kappa\lambda N$. If $\kappa>1$ then $\kappa \mid 4\lambda^2 \Rightarrow \kappa \mid \lambda^2$ wrong by assumption. If $\kappa=1 \Rightarrow \lambda \mid 1 \Rightarrow \lambda=1$ wrong since $R>\sqrt{2}$. The same arguments go through for $\bar{\Delta}$. The representation $(\Delta,\bar{\Delta})$ appears exactly twice as is obvious from the character expansions in (3).

Lemma 2: Let $R>\sqrt{2}$. Then the representation $(\Delta,\bar{\Delta}) = (\frac{1}{2}(\frac{1}{R}+\frac{R}{2})^2\,,\,\frac{1}{2}(\frac{1}{R}-\frac{R}{2})^2)$ which belongs to $\mathbf{Z}(R)$ does not appear in $\mathbf{Z}(R')$ with $R\neq R', R'\geq\sqrt{2}$.

Proof: In order for this representation to appear in $\mathbf{Z}(R')$ there must exist $m,n\in Z$ such that

$$
\begin{array}{ccc}
(\frac{1}{R}+\frac{R}{2})^2 = (\frac{m}{R'}+\frac{nR'}{2})^2 & & \frac{1}{R}+\frac{R}{2} = \frac{m}{R'}+\frac{nR'}{2} \\
 & \Rightarrow & \\
(\frac{1}{R}-\frac{R}{2})^2 = (\frac{m}{R'}-\frac{nR'}{2})^2 & & \frac{1}{R}-\frac{R}{2} = \pm[\frac{m}{R'}-\frac{nR'}{2}]
\end{array}
$$

(i) Choosing the + sign above, $\frac{1}{R}=\frac{m}{R'}\,,\ R=nR' \Rightarrow \mid m \mid = \frac{R'}{R}\,,\ \mid n \mid = \frac{R}{R'}$. But at least one of $\frac{R}{R'}\,,\ \frac{R'}{R}$ is less than one. Thus there is no solution in this case.

(ii) Choosing the $-$ sign, $\frac{2}{R}=nR'\,,\ R=\frac{2m}{R'} \Rightarrow \mid n \mid = \frac{2}{RR'}, \mid m \mid = \frac{RR'}{2}$ since $RR'>2 \Rightarrow \mid n \mid < 1$, thus there is also no solution in this case either. Thus the claim is proven.

Lemma 3: Consider representations with $(\Delta,\bar{\Delta}) = (s^2,0), s=0,1,2,3,4,5$. Then the multiplicity of such representations in $\mathbf{Z}_N$ is $1+2[\frac{s}{N}]$, whereas it is one for any other $\mathbf{Z}(R)$.*

Proof: We can use the character expansions in (3) to find the chiral content (i.e., $(\Delta,0)$ operators). In particular

$$
\mathbf{Z}_N^{\text{chiral}} = \sum_{k=0}^{\infty}\chi_{k^2}\,\bar{\chi}_0 + 2\sum_{\substack{m=1\\k=0}}^{\infty}\chi_{(Nm+k)^2}\bar{\chi}_0 \tag{5}
$$

In this case the multiplicity of $(s^2,0)$ is $1+2[\frac{s}{N}], \forall s\in Z_0^+$ by inspecting (5). Direct inspection in the other cases shows that the multiplicity is one for $s\leq 5$.

⋆ By $[x]$ we mean the integer part of x.

Lemma 4: Let $\mathbf{Z} = \sum_{i=1}^{N} c_i \mathbf{Z}(R_i)$. Then if $R_i \neq \sqrt{2}$, we must have $2c_i \in Z$ in order to have integral multiplicities.

Proof: Let $2c_i \notin Z$. By Lemma 1 and Lemma 2 there is a representation in $\mathbf{Z}(R_i)$ with multiplicity $2c_i$ which does not appear in any other $\mathbf{Z}(R_i), j \neq i$. Thus if $2c_i \neq 0$ this representation will not have an integral multiplicity.

Lemma 5: Let N_s be the number of $(s^2, 0)$ representations present in $\mathbf{Z} = \sum_{i=1}^{N} c_i \mathbf{Z}(R_i)$. If $N_1 = 0$ then $\mathbf{Z}_1$ has to appear with coefficient $c_1 = -\frac{1}{2}$ and all other c_i have to be positive or zero.

Proof: Since $N_1 = 0$ we must express the partition function in terms of Virasoro characters. If $c_i < 0$ for some $i > 1$ then by Lemma 2 there exists at least one Virasoro representation which has a negative multiplicity. In $\mathbf{Z}$ there must be only one $(0,0)$ operator; thus $\sum_{i=1}^{N} c_i = 1$. By assumption the number of $(1,0)$ operators is zero. Since $\mathbf{Z}_1$ contains three of them and any other partition function only one (see Lemma 3), $3c_1 + \sum_{i=2}^{N} c_i = 0$. The two equations imply $c_1 = -\frac{1}{2}$.

Theorem 1: If $N_1 = 0$ then the following possibilities exist for a partition function $\mathbf{Z} = \sum_{i=1}^{N} \mathbf{Z}(R_i)$ such that it contains only one $(0,0)$ operator and all multiplicities are non-negative integers:

$$(a) \quad \mathbf{Z}_{\text{orb}}(R) = \frac{1}{2}(\mathbf{Z}(R) + 2\mathbf{Z}_2 - \mathbf{Z}_1)$$

$$(b) \quad \mathbf{Z}_T = \frac{1}{2}(2\mathbf{Z}_3 + \mathbf{Z}_2 - \mathbf{Z}_1)$$

$$(c) \quad \mathbf{Z}_O = \frac{1}{2}(\mathbf{Z}_4 + \mathbf{Z}_3 + \mathbf{Z}_2 - \mathbf{Z}_1)$$

$$(d) \quad \mathbf{Z}_I = \frac{1}{2}(\mathbf{Z}_5 + \mathbf{Z}_3 + \mathbf{Z}_2 - \mathbf{Z}_1)$$

Proof: Let's write $\mathbf{Z} = -\frac{1}{2}\mathbf{Z}_1 + \sum_{i=2}^{5} c_i \mathbf{Z}_i + \sum_{i=6}^{N} c_i \mathbf{Z}(R_i)$. By Lemma 5 $c_i \geq 0$, $1 = 2, 3 \ldots, N$ and by Lemma 4 $2c_i \in Z_0^+$. By Lemma 3 we obtain the following equations among the c_i and the multiplicities $N_s, s \leq 5$

$$-\frac{1}{2}+c_2+c_3+c_4+c_5+\sum_{i=6}^{N}c_i=1$$

$$-\frac{3}{2}+c_2+c_3+c_4+c_5+\sum_{i=6}^{N}c_i=N_1=0$$

$$-\frac{5}{2}+3c_2+c_3+c_4+c_5+\sum_{i=6}^{N}c_i=N_2 \tag{6}$$

$$-\frac{7}{2}+3c_2+3c_3+c_4+c_5+\sum_{i=6}^{N}c_i=N_3$$

$$-\frac{9}{2}+5c_2+3c_3+3c_4+c_5+\sum_{i=6}^{N}=N_4$$

$$-\frac{11}{2}+5c_2+3c_3+3c_4+3c_5+\sum_{i=6}^{N}c_i=N_5$$

where $N_s \in Z_0^+$ The solution to (6) is:

$$c_2=\frac{N_2+1}{2},\ c_3=\frac{N_3-N_2+1}{2},\ c_4=\frac{N_4-N_3-N_2}{2},$$

$$c_5=\frac{N_5-N_4+1}{2},\ \sum_{i=6}^{N}c_i=-\frac{N_5-N_2}{2} \tag{7}$$

It is obvious that $\sum_{i=2}^{N}c_i=\frac{3}{2}$ and since $c_i\geq 0, i\geq 2, \Rightarrow c_2\leq\frac{3}{2}\Rightarrow N_2+1\leq 3\Rightarrow 0\leq N_2\leq 2$. Thus we need to consider the three cases, $N_2=0,1,2$.

(I) $N_2=0$. Then, $c_2=\frac{1}{2}, c_3=\frac{N_3+1}{2}$, $c_4=\frac{N_4-N_3}{2}$, $c_5=\frac{N_5-N_4+1}{2}$, $\sum_{i=6}^{N}c_i=-\frac{N_5}{2}$

In this case $\sum_{i=3}^{N}c_i=1$, thus either one of the c_i is 1 or two of them are $\frac{1}{2}$.

(Ia) $c_3=1$. This implies $N_3=1, c_4=c_5=0, c_i=0\ i\geq 6\Rightarrow N_4=1, N_5=0$. This solution corresponds to case (b) of Theorem 1. Since $c_3=\frac{N_3+1}{2}\geq\frac{1}{2}$ no other c_i can be equal to 1.

(Ib) $c_3 = c_4 = \frac{1}{2}$, $c_5 = 0 = c_i$ $i \geq 6$, $\Rightarrow N_3 = 0$, $N_4 = 1$, $N_5 = 0$. This solution corresponds to case (c).

(Ic) $c_3 = c_5 = \frac{1}{2}, c_4 = c_i = 0, i \geq 6, \Rightarrow N_3 = 0, N_4 = 0, N_5 = 0$. This solution corresponds to case (d).

(Id) $c_3 = c_i = \frac{1}{2}$ for some $i \geq 6, c_4 = c_5 = 0$. The first three equalities implies $N_3 = N_4 = N_5 = 0$ but $c_5 \neq 0$. Thus no solution exists. The above exhaust all possibilities when $N_2 = 0$.

(II) $N_2 = 1$. Then, $c_2 = 1$, $c_3 = \frac{N_3}{2}$, $c_4 = \frac{N_4 - N_3 - 1}{2}$, $c_5 = \frac{N_5 - N_4 + 1}{2}$, $\sum_{i=6}^{N} c_i = -\frac{N_5 - 1}{2}$. In this case $\sum_{i=3}^{N} c_i = \frac{1}{2} \Rightarrow c_3 \leq \frac{1}{2} \Rightarrow N_3 = 0, 1$. Thus the solution is of the form, one c_i being $\frac{1}{2}$ and the rest being zero.

(IIa) $c_3 = \frac{1}{2}$, $c_4 = c_5 = c_i = 0$, $i \geq 6$, $\Rightarrow N_3 = 1$, $N_4 = 2$, $N_5 = 1$. This corresponds to case (a) with $\mathbf{Z}(R) = \mathbf{Z}_3$.

(IIb) $c_4 = \frac{1}{2}, c_3 = c_5 = c_i = 0, i \geq 6, \Rightarrow N_3 = 0, N_4 = 0, N_5 = 1$. This solution corresponds to case (a) with $\mathbf{Z}(R) = \mathbf{Z}_4$.

(IIc) $c_5 = \frac{1}{2}$, $c_3 = c_4 = c_i$, $i \geq 6$, $\Rightarrow N_3 = 0$, $N_4 = N_5 = 1$. This solution corresponds to case (a) with $\mathbf{Z}(R) = \mathbf{Z}_5$.

(IId) $c_i = \frac{1}{2}$ for some $i \geq 6$, $c_3 = c_4 = c_5 = 0$, $\Rightarrow N_3 = N_5 = 0$, $N_4 = 1$. This solution corresponds to case (a) with $\mathbf{Z}(R)$ other than $\mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_5$. The above exhaust all possibilities with $N_2 = 1$. The last case to consider is $N_2 = 2$.

(III) $N_2 = 2$. Then $c_2 = \frac{3}{2}$, $c_3 = \frac{N_3 - 1}{2}$, $c_4 = \frac{N_4 - N_3 - 2}{2}$, $c_5 = \frac{N_5 - N_4 + 1}{2}$, $\sum_{i=6}^{N} c_i = \frac{2 - N_5}{2}$.

Since $c_i \geq 0$, $i \geq 2$, the only solution in this case is: $c_3 = c_4 = c_5 = c_i = 0$, $i \geq 6 \Rightarrow N_3 = 1$, $N_4 = 3$, $N_5 = 2$. This corresponds to case (a) with $\mathbf{Z}(R) = \mathbf{Z}_2$. The proof of Theorem 1 is now complete.

When $N_1 > 0$, since there is a $\mathcal{U}(1)$ symmetry in the theory one needs to consider the partition functions as sesquilinear forms of the affine $\mathcal{U}(1)$ characters.

Lemma 6: Let $N_1 > 0$. The coefficients c_i of $\mathbf{Z} = \sum_{i=1}^{N} c_i \mathbf{Z}(R_i)$ have the following properties: (1) $c_i \geq 0 \; \forall \, i$ (2) $2c_i \in Z$.

Proof: From Lemma 5 using the same arguments it follows that $c_i \geq 0, i \geq 2$ (that is for any R except $R = \sqrt{2}$). The value of c_1 is given by (6) and is $c_1 = \frac{N_1 - 1}{2}$. Since $N_1 \geq 1 \Rightarrow c_1 \geq 0$.

Thus (1) is proven. It is also obvious that $2c_1 = N_1 - 1 \in Z$. The proof that for $i \geq 2$ $2c_i \in Z$ is exactly the same as the one in Lemma 4.

Theorem 2: When $N_1 > 0$ the only acceptable partition functions $\mathbf{Z} = \sum_{i=1}^{N} c_i \mathbf{Z}(R_i)$ have one of the following two forms.

$$\text{(a) } \mathbf{Z}_t = \mathbf{Z}(R) \quad , \quad \text{(b) } \mathbf{Z}_w = \frac{1}{2}(\mathbf{Z}(R) + \mathbf{Z}(R'))\,,\ R \neq R'$$

Proof: Since c_i are non-negative the only solution to $\sum_{i=1}^{N} c_i = 1$ is either $c_i = 1, c_j = 0, j \neq i$ (case (a)) or $c_i = c_j = \frac{1}{2}, c_k = 0\,,\ k \neq i \neq j \neq k$ (case (b)) q.e.d.

The above complete all the proofs.

There is an extra constraint on partition functions. The operator content must have a consistent operator algebra. In all the partition functions above except $\mathbf{Z}_w$ this is true. It is easy to see that $\mathcal{U}(1)$ invariance implies that $\mathbf{Z}_w$ is unacceptable. There is no operator algebra consistent with additively conserved $\mathcal{U}(1)$ charges. Thus $\mathbf{Z}_w$ does not describe a CFT.

I would like to thank R. Dijkgraaf for lots of discussions.

References

[1] E. Kiritsis, Phys. Lett. B217 (1989) 427.

[2] P. Ginspang, Nucl. Phys. B295[FS21] (1988) 153.

[3] R. Dijkgraaf, E. Verlinde, H. Verlinde, Utrecht preprint, THU-87/30.