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Degenerate Operator Algebra with c = 1, Local SU(2) Invariance and their Realization in the X-Y and A-T Models at Criticality*

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Abstract

The critical O(2) Gaussian and Askin-Teller models are studied. The magnetic operator is identified as the twist field of the scalar field and the electric operator as a particular vertex operator. Invariance under a local SU(2) symmetry is shown to be hidden in the systems above. At a particular point on the critical line ($\lambda = -\frac{\sqrt{2}}{2}$), it is shown that the electric operator generates a degenerate representation of the Virasoro algebra and that it transforms as the fundamental representation under the SU(2) symmetry. The implications are discussed.

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There has been a lot of progress recently in understanding critical phenomena in two dimensions, through conformal field theory and the study of the representations of the Virasoro algebra [1,2].

In this note we intend to add a small piece to the already huge amount of knowledge concerning 2-d critical phenomena. We will identify the electric and magnetic (spin) operators of the Askin-Teller,(A-T), model on the critical line where it is described by a free scalar field which takes values on a circle. In particular we will show that the magnetic operator is the "twist" field that changes the boundary conditions of the scalar field and that its scaling dimension is constant along the critical line. The electric operator will be represented by the vertex operator $:e^{\pm i\phi}:$. Local SU(2) invariance is present on the critical line. We will explicitly construct the currents and a pair of operators that transform in the fundamental of SU(2). At the point $\lambda = -\frac{\sqrt{2}}{2}$ on the $\beta = 1$ critical line we will show that the electric operator transforms non-trivially under the local SU(2) symmetry and that it generates a degenerate representation of the Virasoro algebra. We will also study the consequences of this fact.

The A-T model is described by two Ising spins coupled with a four-spin interaction [3]. There are two couplings, β, governing the strength of the four-spin interaction and λ, governing the spin-spin interactions.† At $\beta = 1$ the strength of the four spin interaction vanishes and there is a line of critical points , $-1 \leq \lambda \leq 1$, of the Kosterlitz-Thouless type with continuously varying critical exponents. The point $\lambda = 0$ corresponds to two independent Ising models whereas at $\lambda = \pm 1$ the model has Z_4 symmetry corresponding to the critical Potts model.

It is well known that the Gaussian model on the critical line is described by a free scalar field which is the phase of the O(2) vector. It is a map : $\phi(\sigma) : S^1 \to S^1$ which is periodic in $\sigma : \phi(\sigma + 2\pi) = \phi(\sigma)$. The operators of this theory can be represented by the so-called vertex operators, $V_a(\sigma) \equiv : e^{ia\phi(\sigma)} :$ and their derivatives. The action in the continuum limit on the critical line $\beta = 1$ is

$$S = -K \int d\tau d\sigma \phi \nabla^2 \phi, \tag{1}$$

where $K = \frac{2}{\pi}(1 - \frac{arccos(\lambda)}{\pi})$ Going to complex coordinates, $ln(z) = \tau + i\sigma$ it is easy to see that

† For more details on the model and its phase diagram we refer the reader to ref. [3]. We will be following the notation of the previous reference.

the theory has a factorized $z, \bar{z}$ dependence.[‡] The two-point function is,

$$\langle 0|\phi(z)\phi(w)|0\rangle = -\frac{1}{4\pi K} ln(z-w) \tag{2}$$

whereas the energy momentum tensor is given by, $T(z) = -\frac{K}{2\pi} : \partial_z\phi(z)\partial_z\phi(z) :$, satisfying the standard O.P.E.,

$$T(z)T(w) = \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} \tag{3}$$

(In the O.P.E. we will always omit the regular pieces). From (3) we derive the Virasoro anomaly for the system, namely $c = 1$. A vertex operator, $V_a(z) \equiv: e^{ia\phi(z)}$: is a primary field with dimension[§] $\Delta_a = \frac{a^2}{8\pi K}$, as shown by the following O.P.E.

$$T(z)V_a(w) = \Delta_a \frac{V_a(w)}{(z-w)^2} + \frac{\partial_w V_a(w)}{(z-w)} \tag{4}$$

K can be related to the thermal critical exponent x_T, as $x_T = \frac{1}{\pi K}$ so that $\Delta_a = a^2 \frac{x_T}{8}$. Since the electric operator is known to have physical dimension $x_p = \frac{x_T}{4}$, the right candidate is $V_{\pm 1} =: e^{\pm i\phi} :$. Its dimension is a function of λ and varies continuously along the critical line.

On the other hand the magnetic critical exponent is known to be $x_H = \frac{1}{8}$ and is constant along the critical line. We will identify the magnetic operator with the twist field, $H^{\pm}(z)$, for the scalar $\phi(z)$. A twisted scalar field is defined to be an anti-periodic instead of a periodic map. On the z-plane this means that the operator $\phi(z)$ is double valued around the position of a twist field. If we place a twist field at $z = 0$, $(\tau = -\infty)$, and another one at $z = \infty$, $(\tau = +\infty)$, they generate a cut on the z-plane that causes the double-valuedness of ϕ. In the presence of twist fields the two-point function of ϕ is modified [4]:

$$\langle 0|\phi(z)\phi(w)|0\rangle_T \equiv \frac{\langle 0|H^{\pm}(\infty)\phi(z)\phi(w)H^{\pm}(0)|0\rangle}{\langle 0|H^{\pm}(\infty)H^{\pm}(0)|0\rangle} = \frac{1}{4\pi K} ln\left(\frac{\sqrt{z}+\sqrt{w}}{\sqrt{z}-\sqrt{w}}\right) \tag{5}$$

We would like to calculate the scaling dimension of the twist field for arbitrary λ. In order

‡ From now on we will concentrate on the z dependence, the total theory being a direct product of the left and right pieces.

§ This is the holomorphic dimension of an operator. There is also the antiholomorphic one, $\bar{\Delta}$. The physical dimension is $\Delta + \bar{\Delta}$ whereas the spin is $\Delta - \bar{\Delta}$.

to do that we have to calculate

$$\langle 0|T(z)|0\rangle_T \equiv \frac{\langle 0|H^{\pm}(\infty)T(z)H^{\pm}(0)|0\rangle}{\langle 0|H^{\pm}(\infty)H^{\pm}(0)|0\rangle}.$$

Using the two-point function, (5), and the definition of the energy-momentum tensor it is straightforward to calculate:

$$\langle 0|T(z)|0\rangle_T = \frac{1}{16}\frac{1}{z^2} \tag{6}$$

Equation (6) gives us two pieces of information. First, that the scaling dimension of the twist field is $\frac{1}{16}$ and second that the twist field is a primary field of the Virasoro algebra (that is it satisfies an O.P.E. of the form depicted in (4)). Thus the physical dimension of the twist field is $\frac{1}{8}$ and it is constant on the $\beta = 1$ line.

It is easy to see that the SU(2) affine algebra is realized in the models on the critical line¶. Let's remind ourselves that the algebra is defined by the current operators $J^a(z), a = 1, 2, 3$ of dimension one, by the following O.P.E.

$$J^a(z)\, J^b(w) = i\epsilon^{abc}\frac{J^c(w)}{(z-w)} + \frac{N}{2}\frac{\delta^{ab}}{(z-w)^2}, \tag{7}$$

where $N = 1, 2, ...$ is the central charge.

The realization is achieved through:

$$J^3(z) = i\sqrt{2\pi K}\partial_z\phi(z) \quad , \quad J^{\pm}(z) = \frac{1}{\sqrt{2}} : e^{\pm i\sqrt{8\pi K}\phi(z)} :\equiv \frac{1}{\sqrt{2}}(J^1(z) \pm iJ^2(z)) \tag{8}$$

and N = 1. The only integrable representation* of the algebra above that can appear is the fundamental [6] which is realized by the $\Delta = \frac{1}{4}$ family ($: e^{\pm i\sqrt{2\pi K}\phi(z)} :$, with isospin $\pm\frac{1}{2}$). Under the semidirect product of the Virasoro algebra and the Kač–Moody algebra, it is a degenerate**representation, the null vector being:

$$[L_{-1} - \frac{\sigma^a}{2}J^a_{-1}]|\frac{1}{4}\rangle_{\pm} \tag{9}$$

σ^a being the Pauli matrices, $\pm$ is the isospin index.

¶ The SU(2) invariance of the action has been first noticed in [5].

* An integrable representation is, in physical terms, a representation whose correlation functions with other operators do not vanish identically

** See ref. [7] for more details.

Due to the degeneracy of the representation the correlation functions satisfy linear differential equations [7]. In our case we have an explicit representation of the operators which makes it possible to compute the correlation functions using the standard formula,

$$\langle 0| \prod_{i=1}^{n} : e^{ia_i\phi(z_i)} : |0\rangle = \prod_{i<j}^{n} (z_{ij})^{\frac{a_i a_j}{4\pi K}} \delta(\sum a_i) \tag{10}$$

Thus if $\phi^i(z)$ is the operator transforming in the fundamental of SU(2), (i=1,2 being the isospin index), then using (10),

$$\langle 0|\phi^i(z_1)\phi^j(z_2)\phi^k(z_3)\phi^l(z_4)|0\rangle = (z_{12}z_{34})^{-\frac{1}{2}}[y(y-1)]^{-\frac{1}{2}} \left[\delta^{ik}\delta^{jl} + (y-1)\delta^{ij}\delta^{kl}\right] \tag{11}$$

where $y = \frac{z_{14}z_{23}}{z_{12}z_{34}}$. The correlation function in (11) does indeed satisfy the differential equation stemming from the degeneracy of the fundamental representation. We should mention that the SU(2) operators are non-local with respect to the electric and magnetic operators.

There is a special value of λ, namely $\lambda = -\frac{\sqrt{2}}{2}$, where the electric operator coincides with the operator transforming with the fundamental of SU(2), both having dimension $\frac{1}{4}$. In this case as we subsequently see, the structure of the Virasoro algebra implies some extra information about the operator content of the theory. Thus we digress a little to develop this point.

There is a special class of representations of the Virasoro algebra called degenerate representations. These are representations with the following property: If one starts from a *hwv* and generates secondary states by applying the lowering operators of the algebra, at some point one obtains a secondary state that has the properties of a *hwv*.

This state is null and orthogonal to the rest of the representation. It generates another representation which is embedded in the initial one. Then the Kač-determinant has a zero corresponding to this null *hwv*. (For more details see [1].)

The Kač-determinant for c = 1 and level m is [2]

$$\det M_m = \prod_{k=1}^{m} \left[\prod_{\substack{r\cdot s=k \\ r\leq s}} \left[\Delta - \frac{(r-s)^2}{4}\right]^{2p(m-k)} \right], \tag{12}$$

where $p(n)$ are the number of partitions of n.

From eq. (4) it is obvious that the Kač-determinant has a zero when $\Delta = \frac{n^2}{4}, n = 0, 1, 2...$ In fact there is an infinity of families embedded in the family with $\Delta = \frac{n^2}{4}$.

Let's consider the representation with the lowest nontrivial dimension, namely $\Delta = \frac{1}{4}$. The Kač-determinant shows that there is a null *hwv* at level two. Its form is given below:

$$|\chi\rangle = [L_2 - L_{-1}^2]|\Delta\rangle. \tag{13}$$

Since the null vector is orthogonal to everything else, it's correlations with other operators will vanish. This fact implies linear differential equations for the correlation functions of $\phi_{1/4}(z)$ with other operators [1].

Consider the n-point function:

$$F_n(z_1, z_2, ..., z) \equiv \langle 0|\phi_1(z_1)\phi_2(z_2)...\phi_{n-1}(z_{n-1})\phi_{1/4}(z)|0\rangle$$

and assume that all $\phi_i(z_1)$ are primary. Then F_n satisfies the following differential equation, due to eq. (13):

$$\left[\sum_{i=1}^{n-1}\left(\frac{1}{z_i}\frac{\partial}{\partial z_i} - \frac{\Delta_i}{z_i^2}\right) + \sum_{i,j=1}^{n-1}\frac{\partial}{\partial z_i}\frac{\partial}{\partial z_j}\right] F_n(z_1, z_2..., = 0, \tag{14}$$

where we shifted $z_i \to z_i - z$, $i = 1, 2..., n-1$.

Let's study the operator products of the $\Delta = \frac{1}{4}$ family with other primary operators. The natural place to look at is the three-point function. The condition that the family $[\Delta_3]$ is contained in the operator product $[\Delta_1] \otimes [\Delta_2]$ is that the three-point function of $[\Delta_1], [\Delta_2], [\Delta_3]$ be non-zero. From projective invariance:

$$\langle 0|\phi_{\Delta_1}(z_1)\phi_{\Delta_2}(z_2)\phi_{\Delta_3}(z_3)|0\rangle = C_{\Delta_1,\Delta_2,\Delta_3} z_{12}^{-\Delta_{12}}\, z_{13}^{-\Delta_{13}}\, z_{23}^{-\Delta_{23}}, \tag{15}$$

where as usual $\Delta_{12} = \Delta_1 + \Delta_2 - \Delta_3$ and so on, while $C_{\Delta_1,\Delta_2,\Delta_3}$ is an overall constant (O.P.E. coefficient) not fixed by projective invariance.

Suppose now that $\Delta_3 = \frac{1}{4}$. Then the three-point function in eq. (15) has to satisfy eq. (14). This implies some constraint on Δ_1, Δ_2 which is:

$$(\Delta_1 - \Delta_2)^2 - \frac{\Delta_1 + \Delta_2}{2} = -\frac{1}{16}. \tag{16}$$

Solving for Δ_1 we obtain $\Delta_1 = (\sqrt{\Delta_2} \pm \frac{1}{2})^2$. If $\Delta_2 = \frac{n^2}{4}$, then $\Delta_1 = \frac{(n\pm1)^2}{4}$. This reminds us of the composition rule for vertex operators. That is the primary field with $\Delta = \frac{n^2}{4}$ can be represented by : $e^{in\phi(z)}$: $n\epsilon\mathbf{Z}$. Then it is easy to evaluate: $[\frac{n^2}{4}] \otimes [\frac{m^2}{4}] \sim [\frac{(n+m)^2}{4}] \oplus [\frac{(n-m)^2}{4}]$ which for $m = 1$ is what eq. (16) implies.

Thus it is obvious that the operator set with $\Delta = \frac{n^2}{4}, n\epsilon\mathbf{Z}$ is closed under O.P.E. due to the O.P.E. for vertex operators.

A non-trivial check that in fact the representation through vertex operators is valid, is obtained by calculating the four-point functions:

$$\langle 0|\phi_{n^2/4}(z_1)\phi_{1/4}(z_2)\phi_{n^2/4}(z_3)\phi_{1/4}(z_4)|0\rangle = z_{24}^{-1/2}\, z_{13}^{-n^2/4} \left[\frac{z_{12}z_{34}}{z_{14}z_{23}}\right]^{n/2}. \tag{17}$$

It is easy to verify that eq. (17) satisfies eq. (14). Thus there is an infinite family of operators associated with the electric operator which form a closed operator algebra.

Now we would like to know what happens if one includes the magnetic operator in the previous set of operators. The operator product of the magnetic and electric operator contains, according to (16), two families, $\left[\frac{1}{16}\right]$ and $\left[\frac{1}{16} + \frac{1}{2}\right]$. By induction it easy to show that we have to include a set of operators, the " twist class", having dimensions $\Delta_n = \frac{(2n+1)^2}{16}$. In fact representing this class of operators by : $e^{\pm i\frac{2n+1}{2}\phi}$:, $n\epsilon Z$, is a valid assumption in the operator set that we are discussing. This fact can be checked again by investigating the respective correlation functions. An important conclusion of the analysis above is that the existence of the electric and magnetic operators implies the existence of the $\left[\frac{1}{16} + \frac{1}{2}\right]$ operator. This is important since at $\lambda = -\frac{\sqrt{2}}{2}$ the model has additional local symmetries (N=1,2 superconformal invariance [8]), and the previous operator is a crucial part of the spectrum. Parafermionic operators that have been found can also be described as above. For example, a parafermionic operator with spin 1/2 and physical dimension 5/8, on the whole $\beta = 1$ line [9] can be described by the family $(\frac{1}{16}, \frac{9}{16})$ belonging to the twist class of operators ($\Delta = \frac{(2n+1)^2}{16}$).

To summarize, using conformal field theory, we analyzed certain aspects of the operator content of the A-T model on the $\beta = 1$ critical line. In particular, we identified the magnetic operator as the twist field for the scalar field by showing that it has the correct dimension independent of λ and the electric operator as a certain vertex operator. We subsequently showed that a local SU(2) symmetry is realized on the critical line by constructing the currents and the fundamental representation. At the special point, $\lambda = -\frac{\sqrt{2}}{2}$, we showed that the electric operator transforms non-trivially under this SU(2) symmetry. By using the fact that the electric operator is a special (degenerate) representation of the Virasoro algebra, we were able to find a whole set of operators required to be present in the spectrum of the theory. Some of the aforementioned results are probably not new to most physicists. However what we want to stress most is the power of conformal field theory and/or algebraic techniques in deducing correlation functions, operator product rules and the operator content of the theory as well as the existence of local symmetries present.

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Note Added

After the completion of this work we received reference [10] where the operator spectrum is discussed on the critical line of the A-T model and reference [11] where issues of modular invariance are discussed for the system above.

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