Waveguides & Applications

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Abstract

In this project we are going to deal with propagation of electromagnetic waves, not in free space, but in the more interesting case, this with boundary conditions. We will examine the cases with rectangular, cylindrical and elliptical geometry. Also we are going to study what is happening inside the surface, i.e. energy losses, skin depth. Finally we will discuss some cases where the theory of wave guides and this of resonant cavities is used.

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1 $\mathcal{I}$ntroduction

Nowdays most of the civil technology is using Maxwell's theory. One of the predictions, is the existence of electromagnetic waves (EM). These waves are not to be concerned in the same way as the common acoustic or sea waves. They have two very important differences. The first is that they are travelling with the largest velocity in the universe (at least locally) and second they don't need "help" to propagate! As known, acoustic(sea) waves propagate because of the air(water) molecules. Well, Maxwell's theory tells us that an EM wave that is created by a space and time modification of the electric and magnetic fields, can even be transmitted in vacuum. The explanation that this waves don't need help to propagate, comes from Maxwell's four equations. Theory tells us that any change of one of the fields, has as a consequence, a change in the other field. Is like the greek prompt[1]:

"*Two hands are better than one*"

1.1 Electrodynamics in vacuum

In physics, when we are trying to solve a problem, i.e. find the energy levels of the Hydrogen atom, we separate the problem in parts of different difficulty levels. In electrodynamics, we begin by considering the static case, of a charge distribution for example, and then we study the dynamic case, where the time dependence is considered.

When we want to study the propagation of EM waves we first see how they propagate in vacuum and then we study the case where propagation takes place, i.e. in a solid. Vacuum is an ideal state of matter in nature, which is defined as the state with no elementary particles nor any bound states of them.

So Maxwell's electrodynamic equations in vacuum are

$$\begin{aligned} \boldsymbol{\nabla} \cdot \boldsymbol{E} &= 0 \\ \boldsymbol{\nabla} \times \boldsymbol{E} &= -\frac{1}{c}\frac{\boldsymbol{\partial B}}{\boldsymbol{\partial t}} \\ \boldsymbol{\nabla} \cdot \boldsymbol{B} &= 0 \\ \boldsymbol{\nabla} \times \boldsymbol{B} &= \frac{1}{c}\frac{\boldsymbol{\partial E}}{\boldsymbol{\partial t}} \end{aligned} \tag{1}$$

[1] In greek :

"*Κράταμε να σε κρατώ,*

να ανεβούμε το βουνό."

Now if we have charge and current distributions, which are responsible for the electromagnetic fields, we have the following equations,

$$\begin{aligned} \boldsymbol{\nabla} \cdot \boldsymbol{E} &= 4\pi\varrho \\ \boldsymbol{\nabla} \times \boldsymbol{E} &= -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial \boldsymbol{t}} \\ \boldsymbol{\nabla} \cdot \boldsymbol{B} &= 0 \\ \boldsymbol{\nabla} \times \boldsymbol{B} &= \frac{4\pi}{c}\boldsymbol{J} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial \boldsymbol{t}} \end{aligned} \tag{2}$$

There is also the equation of continuity

$$\frac{\partial \varrho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{J} = 0$$

which has encoded the charge conservation. In general, first we find the scalar and vector potentials from[2]

$$\Phi(\boldsymbol{r}, t) = \frac{1}{4\pi\varepsilon_o}\int_V \frac{\varrho(\boldsymbol{r}, t_r)}{\tau} dV$$

$$\boldsymbol{A}\ (\boldsymbol{r}, t) = \frac{\mu_o}{4\pi}\int_V \frac{\boldsymbol{J}(\boldsymbol{r}, t_r)}{\tau} dV$$

and then we find the fields from,

$$\begin{aligned} \boldsymbol{E} &= -\boldsymbol{\nabla}\phi - \frac{\partial \boldsymbol{A}}{\partial \boldsymbol{t}} \\ \boldsymbol{B} &= \boldsymbol{\nabla} \times \boldsymbol{A} \end{aligned}$$

The four equations (1) are hiding the information of the existence of electromagnetic waves, with velocity[3,4]

$$c \equiv \frac{1}{\sqrt{\mu_o\varepsilon_o}} = 299,792,458\ m/s$$

[2]In the following equations,inside the integrals, the time dependance has a subscript, r. This time is known as the retarded time. Is a consequence of the finite travelling velocity of light, large enough, but finite. [3, p. 195-216], [2, p. 225,654-657]

[3]Permittivity, , is a physical quantity that describes how an electric field affects and is affected by a dielectric medium, and is determined by the ability of a material to polarize in response to an applied electric field, and thereby to cancel, partially, the field inside the material. Permittivity relates therefore to a materials ability to transmit (or "permit") an electric field. The permittivity of free space (ϵ_0) is $8,85 \cdot 10^{-12}$ F/m. If the medium is isotropic, then is a scalar. Otherwise it is a 3×3 matrix. Permittivity, taken as a function of frequency, can take on real or complex value. In general it is not a constant, as it can vary with the position in the medium, the frequency of the field applied, humidity, temperature and other parameters. In a nonlinear medium, the permittivity can depend upon the strength of the electric field.

[4]In electrodynamics, permeability is the degree of magnetization of a material that responds linearly to an applied magnetic field. Magnetic permeability is represented by the symbol . This term was coined in September, 1885 by Oliver Heaviside.

From equations (1) we have that electromagnetic waves are a superposition of an electric and magnetic field, which are,

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{r},t) &= \boldsymbol{E_o} e^{i(\vec{k}\cdot\vec{r}-\omega t)} \\ \boldsymbol{B}(\boldsymbol{r},t) &= \boldsymbol{B_o} e^{i(\vec{k}\cdot\vec{r}-\omega t)} \end{aligned} \tag{3}$$

Because we don't have boundary conditions, we can show that the electric, the magnetic field and the norm in the direction of propagation are perpendicular to each other.

1.2 Electrodynamics in matter

If we want to study electrodynamics in matter we have to change equations (2). These were the equations that Maxwell wrote in his work, the four equations that unifies electric and magnetic theories. So the equations are,

$$\begin{aligned} \boldsymbol{\nabla}\cdot\boldsymbol{D} &= 4\pi\varrho \\ \boldsymbol{\nabla}\times\boldsymbol{E} &= -\tfrac{1}{c}\tfrac{\partial \boldsymbol{B}}{\partial \boldsymbol{t}} \\ \boldsymbol{\nabla}\cdot\boldsymbol{B} &= 0 \\ \boldsymbol{\nabla}\times\boldsymbol{H} &= \tfrac{4\pi}{c}\boldsymbol{J} + \tfrac{1}{c}\tfrac{\partial \boldsymbol{D}}{\partial \boldsymbol{t}} \end{aligned} \tag{4}$$

where we define,

$$\boldsymbol{D} \equiv \varepsilon_o \boldsymbol{E} + \boldsymbol{\mathcal{P}} \equiv \ \textit{Electric Displacement}$$

$$\boldsymbol{H} \equiv \frac{1}{\mu_o}\boldsymbol{B} + \boldsymbol{\mathcal{M}} \equiv \ Magnetic\ field$$

As in vacuum, the fields satisfy the inhomogeneous wave equations,

$$(\partial_\mu \partial^\mu + k^2)\boldsymbol{A} = 0 \tag{5}$$

where $\boldsymbol{A} \equiv (\boldsymbol{E}, \boldsymbol{B})$.

1.3 Boundary Conditions

Now we will see what is happening to the fields while the EM wave travels via different mediums. The interface in general, will have a charge $\varrho(\boldsymbol{r}, t)$ and current $\boldsymbol{J}(\boldsymbol{r}, t)$ distribution. The divergent form of Maxwell's equation's is,

$$\begin{aligned} \boldsymbol{\nabla}\cdot\boldsymbol{D} &= 4\pi\varrho \\ \boldsymbol{\nabla}\times\boldsymbol{E} &= -\tfrac{1}{c}\tfrac{\partial \boldsymbol{B}}{\partial \boldsymbol{t}} \\ \boldsymbol{\nabla}\cdot\boldsymbol{B} &= 0 \\ \boldsymbol{\nabla}\times\boldsymbol{H} &= \tfrac{4\pi}{c}\boldsymbol{J} + \tfrac{1}{c}\tfrac{\partial \boldsymbol{D}}{\partial \boldsymbol{t}} \end{aligned} \tag{6}$$

Then applying divergent and Stoke's theorems, we find four integral equations which are equivalent of the Maxwell's equations,

$$\oint_S \boldsymbol{D} \cdot \hat{\boldsymbol{n}} da = 4\pi \int_V \varrho dV \tag{7}$$

$$\oint_S \boldsymbol{B} \cdot \hat{\boldsymbol{n}} da = 0 \tag{8}$$

$$\oint_C \boldsymbol{E} \cdot \boldsymbol{dl} = -\frac{1}{c} \int_S \frac{\partial \boldsymbol{D}}{\partial t} \cdot \hat{t} da \tag{9}$$

$$\oint_C \boldsymbol{H} \cdot \boldsymbol{dl} = \int_S \left(\frac{4\pi}{c} \boldsymbol{J} + \frac{1}{c} \frac{\partial \boldsymbol{D}}{\partial t} \right) \cdot \hat{\boldsymbol{t}} da \tag{10}$$

We can let the surface, to move with velocity $\boldsymbol{v}$ [6].

From the first and second integral, we find

$$(\boldsymbol{D_2} - \boldsymbol{D_1}) \cdot \hat{\boldsymbol{n}} = 4\pi\sigma \tag{11}$$

$$(\boldsymbol{B_2} - \boldsymbol{B_1}) \cdot \hat{\boldsymbol{n}} = 0 \tag{12}$$

Now the third integral is

$$\oint_C \boldsymbol{E} \cdot \boldsymbol{dl} = (\hat{\boldsymbol{t}} \times \hat{\boldsymbol{n}}) \cdot (\boldsymbol{E_2} - \boldsymbol{E_1}) \Delta l$$

because $[2, p.21]$

$$I_1 = \int \frac{1}{c} \frac{d\boldsymbol{B}}{dt} \cdot \hat{\boldsymbol{t}} da = 0$$

We finally have

$$\hat{\boldsymbol{t}} \cdot [\hat{\boldsymbol{n}} \times (\boldsymbol{E_2} - \boldsymbol{E_1})] = \hat{\boldsymbol{t}} \cdot [\hat{\boldsymbol{n}} \cdot \boldsymbol{\beta} (\boldsymbol{B_2} - \boldsymbol{B_1})]$$

Similarly we have for the final integral

$$\hat{\boldsymbol{t}} \cdot [\hat{\boldsymbol{n}} \times (\boldsymbol{H_2} - \boldsymbol{H_1}) + \hat{\boldsymbol{n}} \cdot \boldsymbol{\beta} (\boldsymbol{D_2} - \boldsymbol{D_1})] = \frac{4\pi}{c} \boldsymbol{K} \cdot \hat{\boldsymbol{t}}$$

because

$$I_2 = \int \frac{1}{c} \frac{d\boldsymbol{D}}{dt} \cdot \hat{\boldsymbol{t}} da = 0$$

If the interface has no velocity then the boundary conditions become setting $\beta = 0$,

$$\hat{\boldsymbol{n}}\boldsymbol{\cdot}(\boldsymbol{D_2} - \boldsymbol{D_1}) = 4\pi\sigma \tag{13}$$

$$\hat{\boldsymbol{n}}\boldsymbol{\cdot}(\boldsymbol{B_2} - \boldsymbol{B_1}) = 0 \tag{14}$$

$$\hat{\boldsymbol{n}}\times(\boldsymbol{E_2} - \boldsymbol{E_1}) = 0 \tag{15}$$

$$\hat{\boldsymbol{n}}\times(\boldsymbol{H_2} - \boldsymbol{H_1}) = \frac{4\pi}{c}\boldsymbol{K} \tag{16}$$

2 $\mathcal{P}$ropagation in a hollow waveguide

Waveguides can be constructed to carry waves over a wide portion of the EM spectrum, but are especially useful in the microwave and optical frequency ranges[5]. Depending on the frequency, they can be constructed from either conductive or dielectric materials. Waveguides are used for transferring both power and communication signals.

Waveguides used at optical frequencies are dielectric waveguides, structures in which non-conductive material with high permittivity, and this high index of refraction, is surrounded by a material with lower permittivity. The structure guides optical waves by total internal reflection. The most common optical waveguide is optical fiber.

In this part we will consider different geometries for the waveguide, rectangular, cylindrical and elliptical. We are going to derive the cutoff frequencies, for TM and TE modes.

2.1 Rectangular Geometry

Here we are going to derive the TE and TM modes for a rectangular waveguide. We suppose here that the propagation takes place in the x-direction, and that $a \geq b$. Combining the second and the fourth equation of (1) we get the following results

$$\vec{E}(x,y,z,t) = \vec{E}_0(y,z)e^{i(kx-\omega t)}$$

[5]The waveguide concept was first proposed by J.J.Thomson in 1893 and experimentally verified by O.J.Lodge in 1894 using electromagnetic waveguides. The mathematical analysis of the EM propagating modes within a hollow metal cylinder, was first performed by Lord Rayleigh in 1897, [4].

$$\vec{B}(x,y,z,t) = \vec{B}_0(y,z)e^{i(kx-\omega t)}$$

Solving Maxwell's equations (1) we find

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = i\omega B_x$$

$$\frac{\partial E_x}{\partial z} - ikE_z = i\omega B_y$$

$$ikE_y - \frac{\partial E_x}{\partial y} = i\omega B_z$$

$$\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = -\frac{i\omega}{c^2}E_x$$

$$\frac{\partial B_x}{\partial z} - ikB_z = -\frac{i\omega}{c^2}E_y$$

$$ikB_y - \frac{\partial B_x}{\partial y} = -\frac{i\omega}{c^2}E_z$$

Combining the above equations we find

$$E_y = \frac{i}{\left(\frac{\omega}{c}\right)^2 - k^2}\left[k\frac{\partial E_x}{\partial y} + \omega\frac{\partial B_x}{\partial z}\right]$$

$$B_y = \frac{i}{\left(\frac{\omega}{c}\right)^2 - k^2}\left[k\frac{\partial B_x}{\partial y} - \frac{\omega}{c^2}\frac{\partial E_x}{\partial z}\right]$$

$$E_z = \frac{i}{\left(\frac{\omega}{c}\right)^2 - k^2}\left[k\frac{\partial E_x}{\partial z} - \omega\frac{\partial B_x}{\partial y}\right]$$

$$B_z = \frac{i}{\left(\frac{\omega}{c}\right)^2 - k^2}\left[k\frac{\partial B_x}{\partial z} + \frac{\omega}{c^2}\frac{\partial E_x}{\partial y}\right]$$

Combining the above with the remaining equations of (1) we find[6]

$$\left[\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \left(\frac{\omega}{c}\right)^2 - k^2\right]\Psi_x(y,z) = 0 \tag{17}$$

[6] $\Psi_x = (E_x, B_x)$.

- <u>TE - modes :</u> $E_x = 0$

$$\left[\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \left(\frac{\omega}{c}\right)^2 - k^2\right] B_x(y,z) = 0 \tag{18}$$

By seperation of variables

$$B_x = Y(y)Z(z)$$

we conclude

$$\underbrace{\frac{Y''}{Y}}_{-k_y^2} + \underbrace{\frac{Z''}{Z}}_{-k_z^2} + \left(\frac{\omega}{c}\right)^2 - k^2 = 0$$

or

$$Y'' + k_y^2 Y = 0 \tag{19}$$

$$Z'' + k_z^2 Z = 0 \tag{20}$$

where

$$k_y^2 + k_z^2 = \left(\frac{\omega}{c}\right)^2 - k^2 \tag{21}$$

From the boundary condition (13) we have that

$$\begin{aligned} \vec{B}_\perp = 0 \quad &\Longrightarrow \quad \left\{ \begin{array}{l} B_z(z=0) = B_z(z=b) = 0 \\ B_y(y=0) = B_y(y=a) = 0 \end{array} \right\} \Longrightarrow \\ &\Longrightarrow \quad \left\{ \begin{array}{l} \frac{\partial B_x}{\partial z}\Big|_{z=0} = \frac{\partial B_x}{\partial z}\Big|_{z=b} = 0 \\ \frac{\partial B_x}{\partial y}\Big|_{y=0} = \frac{\partial B_x}{\partial y}\Big|_{y=a} = 0 \end{array} \right\} \Longrightarrow \\ &\Longrightarrow \quad \left\{ \begin{array}{l} \frac{dB_x}{dz}\Big|_{z=0,b} = 0 \\ \frac{dB_x}{dy}\Big|_{y=0,a} = 0 \end{array} \right\} \end{aligned} \tag{22}$$

The solutions of the ordinary differential equations (19) and (20) are

$$Y(y) = A_y \cos\frac{m\pi y}{a}, \quad m = 0, \pm1, \pm2, \pm3, \ldots \tag{23}$$

$$Z(z) = A_z \cos\frac{n\pi z}{b}, \quad n = 0, \pm1, \pm2, \pm3, \ldots \tag{24}$$

So the solution is

$$B_{x,mn}(x,y,z,t) = B_0 \cos\frac{m\pi y}{a}\cos\frac{n\pi z}{b}e^{i(kx-\omega t)} \tag{25}$$

From equation (21) we find that

$$k^2 = \frac{\omega^2}{c^2} - k_y^2 - k_z^2 \quad\Rightarrow\quad k_{mn} = \sqrt{\frac{\omega^2}{c^2} - \pi\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)}$$
$$\Rightarrow\quad k_{mn} = \frac{1}{c}\sqrt{\omega^2-\omega_{mn}^2}$$

where we defined

$$\omega_{mn} \equiv c\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}} \tag{26}$$

So we see that the frequencies are quantized. If

$$\omega < \omega_{mn}$$

we don't have propagation of the EM wave[7],

$$k_{mn} = \frac{1}{c}\sqrt{\omega^2-\omega_{mn}^2} = \frac{\omega}{c}\sqrt{1-\frac{\omega_{mn}^2}{\omega^2}} = \frac{\omega}{c}\sqrt{-|\beta_{mn}|} \Rightarrow k_{mn} = \frac{i\omega|\beta_{mn}|}{c}$$

and for that we call the ω_{mn}, cutoff frequencies[8]. The ground cutoff frequency is[9]

$$\omega_{10} = \frac{\pi a}{c}$$

The phase velocity of the wave is

$$v_p \equiv \frac{\omega}{k} = \frac{c\omega}{\sqrt{\omega^2-\omega_{mn}^2}} = \frac{c}{\sqrt{\beta_{mn}}} > c$$

and the group velocity (which sows us, how fast energy is transfered) is[10]

$$v_g \equiv \frac{d\omega}{dk} = \frac{1}{dk/d\omega} = c\sqrt{\beta_{mn}} < c$$

[7] $\beta_{mn} \equiv 1 - \left(\frac{\omega_{mn}}{\omega}\right)^2 < 0$

[8] The fields will be proportional to

$$e^{ik_{mn}x} = e^{\frac{-\omega\beta_{mn}x}{c}}$$

[9] Because $a \geq b$.

[10] The idea of a group velocity, distinct from a wave's phase velocity, was first proposed bt W.R.Hamilton in 1839, and the first full treatment was by Lord Rayleigh in his "Theory of Sound", in 1877.

For completeness the components of the fields are

$$E_x = 0$$

$$E_y = \frac{-i\omega B_0 c^2}{\omega_{mn}^2}\Big(\frac{n\pi}{b}\Big)\cos\frac{m\pi y}{a}\sin\frac{n\pi z}{b}e^{-i(kx-\omega t)}$$

$$E_z = \frac{i\omega B_0 c^2}{\omega_{mn}^2}\Big(\frac{m\pi}{a}\Big)\sin\frac{m\pi y}{a}\cos\frac{n\pi z}{b}e^{-i(kx-\omega t)}$$

$$B_x = B_0\cos\Big(\frac{m\pi y}{a}\Big)\cos\frac{n\pi z}{b}e^{-i(kx-\omega t)}$$

$$B_y = \frac{-ikB_0 c^2}{\omega_{mn}^2}\Big(\frac{m\pi}{a}\Big)\sin\frac{m\pi y}{a}\cos\frac{n\pi z}{b}e^{-i(kx-\omega t)}$$

$$B_z = \frac{-ikB_0 c^2}{\omega_{mn}^2}\Big(\frac{n\pi}{b}\Big)\cos\frac{m\pi y}{a}\sin\frac{n\pi z}{b}e^{-i(kx-\omega t)}$$

- <u>TM - modes :</u> $B_x = 0$

In the same way, we find

$$E_x = E_0\sin\frac{m\pi y}{a}\sin\frac{n\pi z}{b}e^{-i(kx-\omega t)}$$

$$E_y = \frac{-ikE_0 c^2}{\omega_{mn}^2}\Big(\frac{m\pi}{a}\Big)\cos\frac{m\pi y}{a}\sin\frac{n\pi z}{b}e^{-i(kx-\omega t)}$$

$$E_z = \frac{ikE_0 c^2}{\omega_{mn}^2}\Big(\frac{n\pi}{b}\Big)\sin\frac{m\pi y}{a}\cos\frac{n\pi z}{b}e^{-i(kx-\omega t)}$$

$$B_x = 0$$

$$B_y = \frac{i\omega E_0 c^2}{\omega_{mn}^2}\Big(\frac{m\pi}{a}\Big)\sin\frac{m\pi y}{a}\cos\frac{n\pi z}{b}e^{-i(kx-\omega t)}$$

$$B_z = \frac{i\omega E_0 c^2}{\omega_{mn}^2}\Big(\frac{m\pi}{a}\Big)\cos\frac{m\pi y}{a}\sin\frac{n\pi z}{b}e^{-i(kx-\omega t)}$$

2.2 Cylindrical Geometry

Now are are going to do the same thing for a cylindrical waveguide. Here the propagation takes place in the z-direction. Combining the second and the fourth equation of (1) we get the following results

$$\vec{E}(r,\theta,z,t) = \vec{E}_0(r,\theta)e^{i(kz-\omega t)}$$

$$\vec{B}(r,\theta,z,t) = \vec{B}_0(r,\theta)e^{i(kz-\omega t)}$$

From Maxwell's equations (1) we find,

$$\frac{1}{r}\frac{\partial E_z}{\partial \theta} - ikE_\theta = i\omega B_r$$

$$ikE_r - \frac{\partial E_z}{\partial r} = i\omega B_\theta$$

$$\frac{1}{r}\left(\frac{\partial}{\partial r}(rE_\theta) - \frac{\partial E_r}{\partial \theta}\right) = i\omega B_z$$

$$\frac{1}{r}\frac{\partial B_z}{\partial \theta} - ikB_\theta = -i\frac{\omega}{c^2}E_r$$

$$ikB_r - \frac{\partial B_z}{\partial r} = -i\frac{\omega}{c^2}E_\theta$$

$$\frac{1}{r}\left(\frac{\partial (rB_\theta)}{\partial r} - \frac{\partial B_r}{\partial \theta}\right) = -i\frac{\omega}{c^2}E_z$$

Combining the above equations we find

$$E_\theta = \frac{ik}{(\frac{\omega}{c})^2 - k^2}\Big(\frac{1}{r}\frac{\partial E_z}{\partial \theta} - \frac{\omega}{k}\frac{\partial B_z}{\partial r}\Big)$$

$$B_r = \frac{-i}{(\frac{\omega}{c})^2 - k^2}\big(\frac{\omega}{rc^2}\frac{\partial E_z}{\partial \theta} - k\frac{\partial B_z}{\partial r}\big)$$

$$B_\theta = \frac{ik}{(\frac{\omega}{c})^2 - k^2}\big(\frac{1}{r}\frac{\partial B_z}{\partial \theta} + \frac{\omega}{kc^2}\frac{\partial E_z}{\partial r}\big)$$

$$E_r = \frac{i}{(\frac{\omega}{c})^2 - k^2}\big(\frac{\omega}{r}\frac{\partial B_z}{\partial \theta} + k\frac{\partial E_z}{\partial r}\big)$$

Combining the above with the remaining equations of (1) we find

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \left(\frac{\omega}{c}\right)^2 - k^2\right)\Psi_z = 0 \tag{27}$$

- <u>TE - modes :</u> $E_z = 0$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \left(\frac{\omega}{c}\right)^2 - k^2\right)B_z = 0 \tag{28}$$

By seperation of variables

$$B_z(r,\theta) = R(r)\Theta(\theta)$$

we conclude

$$\Phi'' = -m^2\Phi$$

and

$$r^2R'' + rR' + \left[r^2\Big(\frac{\omega^2}{c^2} - k^2\Big) - m^2\right] = 0$$

The general solutions are[11]

$$\Theta(\theta) = \left\{\begin{array}{c} e^{im\theta} \\ e^{-im\theta} \end{array}\right\}$$

[11] $n^2 \equiv \frac{\omega^2}{c^2} - k^2$

and[12]

$$R(nr) = \left\{ \begin{array}{c} J_m(nr) \\ N_m(nr) \end{array} \right\}$$

So, the general solution is

$$B_{z,m}(r,\theta,z,t) = \Big(A_r J_m(nr) + B_r N_m(nr)\Big) \cdot \Big(A_\theta e^{im\theta} + B_\theta e^{-im\theta}\Big) \cdot e^{-i(kz-\omega t)}$$

The boundary conditions in this problem are:
1) the fields are not infinite in r=R,
2) the field have the same value for angles and +2,
3) $E_{//}(R,\theta) = 0$,
4) $B_\perp(R,\theta) = 0$.

We must say that we can have only TE or TM modes. From the first boundary condition we have that $B_r = 0$. Now in r=R and for any we have $E_z(R) = 0$. So the condition is

$$J_m(nr) = 0$$

From the above equation, because R is fixed, we see that only some values of n are permitted and these values are the roots of J_m. On the other hand we have that

$$\left.\frac{\partial B_z}{\partial r}\right|_{r=R} = 0, \ \forall\ \theta.$$

This condition is satisfied if

$$J'_m(nR) = 0$$

From the last two conditions we see that either

$$E_z \neq 0 \quad and \quad B_z = 0$$

or

$$B_z \neq 0 \quad and \quad E_z = 0$$

can exist. So we conclude that the solution is

$$B_z(r,\theta,z,t) = \sum_m B_{z,m}(r,\theta,z,t) = \sum_m J_m(nr)\Big(Ae^{im\theta} + Be^{-im\theta}\Big) \cdot e^{-i(kz-\omega t)}$$

where A and B are in general complex numbers. So the components are

[12]Here $J_m(x)$ are the Bessel equations of the first kind and integral order m and $N_m(x)$ are the Bessel equations of the second kind and integral order m or Neumann equations of m order. The second solutions ($N_m(x)$), where $x = nr$, goes to infinity as r goes to 0, so we keep only the first solution, $J_m(x)$.

$$E_r = \frac{m\omega}{k_c^2 r} B_0 J_m(nr)(Ae^{im\theta} - Be^{-im\theta})e^{-i(kz-\omega t)}$$

$$E_\theta = \frac{i\omega B_0 J'_m(nr)}{k_c^2}(Ae^{im\theta} + Be^{-im\theta})e^{-i(kz-\omega t)}$$

$$E_z = 0$$

$$B_r = \frac{-ikB_0 J'_m(nr)}{k_c^2}(Ae^{im\theta} + Be^{-im\theta})e^{-i(kz-\omega t)}$$

$$B_\theta = \frac{kmB_0 J_m(nr)}{k_c^2 r}(Ae^{im\theta} - Be^{-im\theta})e^{-i(kz-\omega t)}$$

$$B_z = B_0 J_m(nr)(Ae^{im\theta} + Be^{-im\theta})e^{-i(kz-\omega t)}$$

- TM - modes : $B_z = 0$

In the same way, the components for the TM modes are

$$E_r = \frac{-ikE_0 J'_m(nr)}{k_c^2}(Ae^{im\theta} + Be^{-im\theta})e^{-i(kz-\omega t)}$$

$$E_\theta = \frac{mkE_0 J_m(nr)}{k_c^2 r}(Ae^{im\theta} - Be^{-im\theta})e^{-i(kz-\omega t)}$$

$$E_z = E_0 J_m(nr)(Ae^{im\theta} + Be^{-im\theta})e^{-i(kz-\omega t)}$$

$$B_r = \frac{-m\omega E_0 J_m(nr)}{k_c^2 r}(Ae^{im\theta} + Be^{-im\theta})e^{-i(kz-\omega t)}$$

$$B_\theta = \frac{-i\omega E_0 J'_m(nr)}{k_c^2}(Ae^{im\theta} - Be^{-im\theta})e^{-i(kz-\omega t)}$$

$$B_z = 0$$

As we said before, A and B are not fixed. There are some interesting cases, where

$$B = \pm A$$

or

$$A = 0 \; or \; B = 0.$$

for each mode. In the first case, the expotential part gives us a sine or a cosine. In the second case, where one of the numbers is vanishing, each field depends on by a factor of $e^{\pm im\theta}$. This field can be represented us a superposition of two different waves, for example we will have for the z-component in TM modes,

$$E_z = E_0 J_m(nr)(\cos m\theta \pm i \sin m\theta)e^{-i(kz-\omega t)}.$$

Then by setting $i \equiv e^{\pm \frac{i\pi}{2}}$, we see that the sine wave has a phase difference of /2 from the cosine wave. But we know that a wave that is a superposition of two plane waves, which have local and time phase difference equal to 90^0, has spiral polarization. So to conclude, the waves that instead of sines and cosines, depend to by a factor of $e^{\pm im\theta}$, are cyclic polarized to the z axes and are called spiral polarized waves.

2.3 Elliprical Geometry

Now are are going to do the same thing for an elliptical waveguide. Here the propagation also takes place in the z-direction. The equation that we have to solve now is

$$\left(\nabla^2 + k^2\right)\psi_z = 0 \Rightarrow \left[\frac{1}{\sinh^2 r + \sin^2\theta}\left(\frac{\partial^2}{\partial^2 r} + \frac{\partial^2}{\partial\theta^2}\right) + \frac{\partial^2}{\partial z^2} + k^2\right]\psi_z = 0$$

If we separate the variables

$$\psi_z = \psi_z(r,\theta,z) = R(r)\Theta(\theta)Z(z)$$

we have

$$\underbrace{\frac{1}{\sinh^2 r + \sin^2\theta}\left(\frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}\right)}_{=m^2} + \underbrace{\frac{1}{Z}\frac{d^2Z}{dz^2}}_{=-m^2} + k^2 = 0 \Rightarrow$$

$$\Rightarrow \left\{\begin{array}{c} \frac{Z''}{Z} = -(k^2+m^2) \Rightarrow Z'' = -(k^2+m^2)Z = -\phi_{km}^2 Z \\ \frac{1}{\sinh^2 r + \sin^2\theta}\left(\frac{R''}{R} + \frac{\Theta''}{\Theta}\right) = m^2 \Rightarrow \underbrace{\left(\frac{R''}{R} - m^2\sinh^2 r\right)}_{=b} + \underbrace{\left(\frac{\Theta''}{\Theta} - m^2\sin^2\theta\right)}_{=-b} = 0 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left\{ \begin{array}{c} Z'' = -\phi_{km}^2 Z \\ \frac{R''}{R} - m^2 \sinh^2 r = b \Rightarrow R'' - (b + m^2 \sinh^2 r)R = 0 \\ \frac{\Theta''}{\Theta} - m^2 \sin^2 \theta = -b \Rightarrow \Theta'' + (b - m^2 \sin^2 \theta)\Theta = 0 \end{array} \right\} \Rightarrow$$

$$\Rightarrow \left\{ \begin{array}{c} Z'' = -\phi_{km}^2 Z \\ R'' - (b + \frac{1}{2}m^2[\cosh(2r) - 1])R = 0 \Rightarrow R'' - \Big[(b - \frac{1}{2}m^2) + \frac{1}{2}m^2 \cosh(2r)\Big]R = 0 \\ \Theta'' + (b - \frac{1}{2}m^2[1 - \cos(2\theta)])R = 0 \Rightarrow \Theta'' - \Big[(b + \frac{1}{2}m^2) + \frac{1}{2}m^2 \cos(2\theta)\Big]\Theta = 0 \end{array} \right\}$$

Now if we set

$$p \equiv b - \frac{m^2}{2}, \quad q \equiv -\frac{m^2}{4}, \quad \sinh^2 r = \frac{1}{2}[\cosh(2r) - 1], \quad \sin^2 \theta = \frac{1}{2}[1 - \cos(2\theta)]$$

we have

$$Z'' = -\phi_{km}^2 Z$$

$$R'' - (p - 2q \cosh 2r)R = 0, \quad (modified\ \ Mathieu\ \ differential\ \ equation)$$

$$\Theta'' + (p - 2q \cos 2\theta)\Theta = 0, \quad (Mathieu\ \ differential\ \ equation)$$

with solutions[13] [12] [4]

$$Z(z) = A_z e^{i\phi_{km} z} + B_z e^{-i\phi_{km} z}$$

$$R(r) = A_r C(p, q, -ir) + B_r S(p, q, -ir)$$

$$\Theta(\theta) = A_\theta C(p, q, \theta) + B_\theta S(p, q, \theta)$$

The difference with the cylindrical problem is that because the solutions are Mathieu functions, which are even and odd, we are going to write them for example ${}_eTM_{01}$ or ${}_oTM_{01}$.

[13]Here C(p,q,x) and S(p,q,x) are Mathieu's even and odd functions. These functions appear in physical problems, involving elliptical shapes and were first introduced by Mathieu in 1868, when analyzing the motion of elliptical membranes.

3 $\mathcal{S}$kin $\mathcal{D}$epth and $\mathcal{E}$nergy $\mathcal{L}$osses $\mathcal{I}$nside a $\mathcal{W}$aveguide

In the last section we saw the modes of the fields in three different geometries. In this section, we will see what is happening inside, the lateral surface of the waveguide. To be more specific, we will study the skin depth of the wave guides and the energy losses of the waves that are passing through the interface.

3.1 Skin Depth

If we want to call ourselves realists, we have to admit that equations

$$\boldsymbol{E} = \boldsymbol{E_0} e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$\boldsymbol{B} = \boldsymbol{B_0} e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

with $\boldsymbol{k}$ real, is not correct when the EM wave is travelling inside matter. In these equations, permeability () and permittivity () are assumed to be independent of frequency. This is the reason that EM waves have constant velocity of propagation in vacuum, but in reality, all media show some dispersion. For most media, we can consider the permeability equal to this of vacuum,

$$\mu \simeq 1$$

To find the relation between permittivity and frequency we need to develop a simple model of dispersion. The relation is $[2, p.284-298]$, $[3, p.145-155]$

$$\epsilon(\omega) = \epsilon_0 + i\frac{4\pi N e^2 f_0}{m\omega(\gamma_0 - i\omega)} = \epsilon_0 + i\frac{4\pi\sigma}{\omega}$$

where we define the conductivity

$$\sigma \equiv \frac{N e^2 f_0}{m\omega(\gamma_0 - i\omega)}$$

If the above relation stands, then we have also to change $\vec{k}$, which is given by the equation

$$k = \frac{\omega}{v} = \sqrt{\mu\epsilon(\omega)}\frac{\omega}{c} \Rightarrow k^2 = \mu\epsilon(\omega)\frac{\omega^2}{c^2} = \mu\epsilon_0\frac{\omega^2}{c^2}\Big(1 + i\frac{4\pi\sigma}{\omega\epsilon_0}\Big) \in C$$

We see that k is a complex number. So we can in general write it as

$$k = \beta + i\frac{\alpha}{2} \Rightarrow k^2 = \Big(\beta^2 - \frac{\alpha^2}{4}\Big) + i\beta\alpha = \mu\epsilon\frac{\omega^2}{c^2} + i\mu\epsilon\frac{\omega^2}{c^2}\frac{4\pi\sigma}{\omega\epsilon}$$

The only thing we have to do now is to find the real and imaginary part of k. After some algebra, the solutions are

$$\frac{\alpha}{2} = \sqrt{\mu\epsilon}\frac{\omega}{c}\sqrt{\frac{\sqrt{1+(\frac{4\pi\sigma}{\omega\epsilon})^2}-1}{2}}$$

$$\beta = \sqrt{\mu\epsilon}\frac{\omega}{c}\sqrt{\frac{\sqrt{1+\left(\frac{4\pi\sigma}{\omega\epsilon}\right)^2}+1}{2}}$$

For poor conductors, we have

$$\frac{4\pi\sigma}{\omega\epsilon} << 1$$

and k approximately becomes

$$k = \beta + i\frac{\alpha}{2} \simeq \sqrt{\mu\epsilon}\frac{\omega}{c} + i\frac{2\pi}{c}\sqrt{\frac{\mu}{\epsilon}}\sigma$$

correct to first order in $\sigma/\omega\epsilon$. In this case, the attenuation of the wave, the imaginary part, is independent of frequency. On the other hand, for a good conductor

$$\frac{4\pi\sigma}{\omega\epsilon} >> 1$$

we have that the real and the imaginary part of k are approximately (in first order in $\omega\epsilon/\sigma$) equal,

$$k \simeq (1+i)\frac{\sqrt{2\pi\omega\mu\sigma}}{c}$$

So we see, that in good conductors EM are proportional to

$$(\vec{E},\vec{B}) \sim e^{i(\vec{k}\cdot\vec{r}-\omega t)} = \underbrace{e^{-\frac{\alpha}{2}\hat{n}\cdot\vec{r}}}_{attenuation\ part} \cdot \underbrace{e^{i(\beta\hat{n}\cdot\vec{r}-\omega t)}}_{propagation\ part}$$

Because of the attenuation part we can define the penetration depth of the conductor considered,

$$\delta \equiv \frac{2}{\alpha} \simeq \frac{c}{\sqrt{2\pi\omega\mu\sigma}}, \quad (skin\ depth)$$

Just to see for what numbers we are speaking of,

Frequency	*Skin Depth*
60 Hz	8,57 mm
10 kHz	$0,66\ mm$
100 kHz	$0,21\ mm$
1 MHz	$66\ \mu m$
10 MHz	$21\ \mu m$
110 MHz	$7,1\ \mu m$

3.2 Energy Losses

Until now we have seen what is happening in the hollow part of the waveguide. Here we are going to study the energy losses inside the boundary, conductive surface. The boundary conditions are

$$\hat{\boldsymbol{n}}\cdot\boldsymbol{D} = 4\pi\Sigma$$

$$\hat{\boldsymbol{n}}\times\boldsymbol{H} = \frac{4\pi}{c}\boldsymbol{K}$$

$$\hat{\boldsymbol{n}}\times(\boldsymbol{E} - \boldsymbol{E_c}) = 0$$

$$\hat{\boldsymbol{n}}\cdot(\boldsymbol{B} - \boldsymbol{B_c}) = 0$$

Maxwell's equations give's us

$$\boldsymbol{\nabla} \times \boldsymbol{E_c} = -\frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t} = \frac{i\omega}{c}\boldsymbol{H_c} \Rightarrow \boldsymbol{H_c} = \frac{-ic}{\mu_c\omega}\boldsymbol{\nabla} \times \boldsymbol{E_c}$$

and[14]

$$\boldsymbol{\nabla} \times \boldsymbol{H_c} = \frac{4\pi}{c}\boldsymbol{J} + \frac{1}{c}\underbrace{\frac{\partial \boldsymbol{D}}{\partial t}}_{\simeq 0} \simeq \frac{4\pi\sigma}{c}\boldsymbol{E_c} \Rightarrow \boldsymbol{E_c} \simeq \frac{c}{4\pi\sigma}\boldsymbol{\nabla} \times \boldsymbol{H_c}$$

Now by substituting

$$\boldsymbol{H_c} = \boldsymbol{H}_{//}e^{-i\omega t}$$

[14]Here we used Ohm's equation

$$\boldsymbol{J} = \sigma\boldsymbol{E}$$

and

$$\boldsymbol{\nabla} \simeq -\hat{\boldsymbol{n}}\frac{\partial}{\partial \xi}$$

we have

$$\boldsymbol{E_c} \simeq -\frac{c}{4\pi\sigma}\hat{\boldsymbol{n}}\times\frac{\partial \boldsymbol{H_c}}{\partial \xi} \Rightarrow \frac{\partial \boldsymbol{E_c}}{\partial \xi} = -\frac{c}{4\pi\sigma}\frac{\partial^2}{\partial \xi^2}\Big(\hat{\boldsymbol{n}} \times \boldsymbol{H_c}\Big)$$

and

$$\boldsymbol{H_c} = \frac{ic}{\mu_v \omega}\hat{\boldsymbol{n}}\times\frac{\partial \boldsymbol{E_c}}{\partial \xi}$$

From the last equation we see that

$$\hat{\boldsymbol{n}} \cdot \boldsymbol{H_c} = \frac{ic}{\mu_c \omega}\hat{\boldsymbol{n}}\cdot\left(\hat{\boldsymbol{n}}\times\frac{\partial \boldsymbol{E_c}}{\partial \xi}\right) = 0 \tag{29}$$

and

$$\hat{\boldsymbol{n}} \times \boldsymbol{H_c} = -\frac{ic^2}{4\pi\mu_c\omega\sigma}\hat{\boldsymbol{n}}\times\left[\hat{\boldsymbol{n}}\times\frac{\partial^2}{\partial \xi^2}\Big(\hat{\boldsymbol{n}} \times \boldsymbol{H_c}\Big)\right] \Rightarrow$$

$$\Rightarrow \hat{\boldsymbol{n}}\times\left[\hat{\boldsymbol{n}}\times\frac{\partial^2}{\partial \xi^2}\Big(\hat{\boldsymbol{n}} \times \boldsymbol{H_c}\Big)\right] = \frac{i4\pi\mu_c\omega\sigma}{c^2}\Big(\hat{\boldsymbol{n}} \times \boldsymbol{H_c}\Big) = -\frac{\partial^2}{\partial \xi^2}\Big(\hat{\boldsymbol{n}} \times \boldsymbol{H_c}\Big) \Rightarrow$$

$$\Rightarrow \frac{\partial^2}{\partial \xi^2}\Big(\hat{\boldsymbol{n}} \times \boldsymbol{H_c}\Big) + \frac{i4\pi\mu_c\omega\sigma}{c^2}\Big(\hat{\boldsymbol{n}} \times \boldsymbol{H_c}\Big) = 0 \Rightarrow$$

$$\Rightarrow \frac{\partial^2}{\partial \xi^2}\Big(\hat{\boldsymbol{n}} \times \boldsymbol{H_c}\Big) + \frac{2i}{\delta}\Big(\hat{\boldsymbol{n}} \times \boldsymbol{H_c}\Big) = 0 \tag{30}$$

where

$$\delta \equiv \frac{c}{\sqrt{4\pi}}\sqrt{\frac{2}{\mu_c\omega\sigma}}$$

which is the skin depth of the conductor. The solution of (30) is

$$\boldsymbol{H_c} = \boldsymbol{H_{//}}\ e^{-\xi/\delta}\ e^{i\xi/\delta} \tag{31}$$

and from that we find the electric field

$$\frac{\partial \boldsymbol{H_c}}{\partial \xi} = -\frac{(1-i)}{\delta} \boldsymbol{H}_{//}\ e^{-\xi/\delta}\ e^{i\xi/\delta} \Rightarrow$$

$$\Rightarrow \boldsymbol{E_c} \simeq \sqrt{\frac{\mu_c \omega}{8\pi\sigma}}(1-i)\Big(\hat{\boldsymbol{n}} \times \boldsymbol{H}_{//}\Big)\ e^{-(1-i)\xi/\delta} \tag{32}$$

The existence of a small tangential component of $\boldsymbol{E}$ outside the surface, in addition to the normal $\boldsymbol{E}$ and tangential $\boldsymbol{H}$, means that there is a power flow into the conductor. The time-average power absorbed per unit area is

$$\frac{dP_{loss}}{dS} = -Re\Big[\hat{\boldsymbol{n}} \cdot \boldsymbol{S}\Big] = -\frac{c}{8\pi} Re\Big[\hat{\boldsymbol{n}} \cdot \Big(\boldsymbol{E} \times \boldsymbol{H}^*\Big)\Big] = \frac{1}{4\pi}\frac{\mu_c \omega \delta}{4}\Big|\boldsymbol{H}_{//}\Big|^2$$

This result can be given a simple interpretation as ohmic losses in the body of the conductor. The above equation will allow us to calculate approximately the resistive losses for practical cavities, transmission lines, and waveguides, provided we have solved for the fields in the idealized problem of infinite conductivity.

4 $\mathcal{A}$pplications

Until now, we have seen the basic theory of waveguides. Naturally, there are many applications for this part of electrodynamic theory, from transferring a huge amount of information through fiber optics to understanding sleep patterns and relaxation levels of human beings.

Because of the rotation symmetry, of a cylindrical waveguide, it is allowed a rotation of the polarization plane round the z-axis. The above application is very useful in many optical experiments.

Waveguides with elliptical geometry, are useful to know, because there are some cases where waveguides with cyclic cross section distort and their cross section becomes elliptic.

On the other hand, it is technically useful to know the value of skin depth for the conductor that we use, because we may want to allow a magnetic field to pass through a conductive surface, but in same the time we don't want to lose the modes inside the waveguide.

One of the most important applications are the so called resonant cavities. The basics of this theory is the same with this of waveguides, which we have analyzed above, with only one difference. We add two more boundary conditions, by inserting two end surfaces vertical to the axis, so that we have stationary solutions. Resonant cavities have many applications, with most used, this of accelerators and this of the model where we take the Earth-ionosphere system as a spherical resonant cavity. We will discuss these two applications in the following subsections.

4.1 Resonant Cavities and Accelerators

In accelerators, resonant cavities are used to accelerate the bundles of particles. The idea is simple and is the same for linear and cyclic accelerators (topically a cyclic accelerator can be considered as linear). What we have to do is to apply an electric field in the resonant cavity with such a frequency, so that when the bundle enters the cavity will accelerate and not decelerate.

As an application we are now going to study resonant cavities produced by cylindrical waveguides with end surfaces vertical to the axis of the cylinder. We have infinite conductivity at the cavity walls and a dielectric material inside a cylinder of length d.

So we get field components that of standing waves on the z axis

$$A\sin kz + B\cos kz$$

The boundary conditions for the cylinder of length d gives us that

$$k = n\frac{\pi}{d}, \quad n = 0, 1, 2, \ldots$$

- <u>TM modes:</u> $H_z = 0$

As

$$\vec{E}_t = 0$$

on the boundary surfaces z=0, z=d the solutions of $\vec{E}_z$ are

$$E_z = E_0(x, y)\cos\frac{n\pi z}{d}, \quad n = 0, 1, 2, \ldots$$

- <u>TE modes:</u> $E_z = 0$

The magnetic field has solutions

$$H_z = H_0(x, y)\sin\frac{n\pi z}{d}, \quad n = 1, 2, 3, \ldots$$

Combining the

$$H_t = \frac{\pm 1}{Z}\vec{k} \times \vec{E}_t$$

where Z is the wave impendance of the conductor

$$Z = \left\{ \begin{array}{ccc} \frac{k}{k_0}\sqrt{\frac{\mu}{\epsilon}} & , & TM \\ \frac{k_0}{k}\frac{\mu}{\epsilon} & , & TE \end{array} \right\}$$

with

$$TM : \quad E_t = \pm\frac{ik}{\gamma^2}\vec{\nabla}_t\psi$$

$$TE : \quad H_t = \pm\frac{ik}{\gamma^2}\vec{\nabla}_t\psi$$

we get the equations for the transverse fields

$$\left\{ \begin{array}{c} \vec{E}_t = -\frac{n\pi}{\gamma^2 d}\sin\frac{n\pi z}{d}\vec{\nabla}_t\psi \\ \vec{H}_t = \frac{i\epsilon\omega}{\gamma^2 c}\cos\frac{n\pi z}{d}\hat{k} \times \vec{\nabla}_t\psi \end{array} \right\}, \quad TM \tag{33}$$

$$\left\{ \begin{array}{c} \vec{E}_t = -\frac{i\omega\mu}{\gamma^2 c}\sin\frac{n\pi z}{d}\hat{k}\times\vec{\nabla}_t\psi \\ \vec{H}_t = \frac{n\pi}{\gamma^2 d}\cos\frac{n\pi z}{d}\vec{\nabla}_t\psi \end{array} \right\}, \quad TE \tag{34}$$

where

$$\gamma^2 = \mu\epsilon\frac{\omega^2}{c^2} - \Big(\frac{n\pi}{d}\Big)^2$$

For each n we get a different γ_λ which gives us an eigenfrequency

$$\omega_{\lambda n}^2 = \frac{c^2}{\mu\epsilon}\Big[\gamma_\lambda^2 + \Big(\frac{n\pi}{d}\Big)^2\Big]$$

The resonance frequencies can be determined graphically on the figure of axial wave number k versus frequency in a waveguide by demanding that

$$k = \frac{n\pi}{d}.$$

It is usually expedient to choose the various dimensions of the cavity so that the resonant frequency of operation lies well separated from other resonant frequencies. Then the cavity will be stable in operation and insensitive to perturbing effects associated with frequency drifts, changes in loading, etc. For example we can allow the frequencies to change by applying a piston in a cylindrical resonant cavity so that by changing the height. The field inside the resonant cavity will be

$$\psi = E(\rho,\phi) = E_0 J_m(\gamma_{ml}\rho)e^{\pm im\phi}$$

where

$$\gamma_{mn} = \frac{x_{ml}}{R}$$

and x_{mn} is the lth root of the equation, $J_m(x) = 0$. The integers m and l take on the values $m = 0, 1, 2, \ldots$ and $l = 1, 2, 3, \ldots$. The first three values are

$$\begin{array}{lcl} m = 0, \;\; x_{0l} & = & 2.405,\, 5.520,\, 8.654,\, \ldots \\ m = 1, \;\; x_{1l} & = & 3.832,\, 7.016,\, 10.173,\, \ldots \\ m = 2, \;\; x_{2l} & = & 5.136,\, 8.417,\, 11.620,\, \ldots \end{array}$$

For higher roots, the asymptotic formula is

$$x_{ml} \simeq l\pi + \Big(m - \frac{1}{2}\Big)\frac{\pi}{2}$$

The resonance frequencies are given by

$$\omega_{mln} = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\frac{x_{ml}^2}{R^2} + \frac{n^2\pi^2}{d^2}}$$

The lowest TM mode is TM_{010} with resonance frequency

$$\omega_{010} = \frac{2,405}{\sqrt{\mu\epsilon}}\frac{c}{R}$$

The components are

$$E_z = E_0 J_0\Big(\frac{2,405\rho}{R}\Big)e^{-i\omega t}$$

$$H_\phi = -i\sqrt{\frac{\epsilon}{\mu}}E_0 J_1\Big(\frac{2,405\rho}{R}\Big)e^{-i\omega t}$$

For TE modes the difference is that now from the boundary condition we have

$$\gamma_{ml} = \frac{x'_{ml}}{R}$$

where x'_{ml} is the lth root of $J'_m(x) = 0$. Some values are

$$\begin{array}{lcl} m=0,\ \ x'_{0l} & = & 3.832,\ 7.016,\ 10.173,\ \ldots \\ m=1,\ \ x'_{1l} & = & 1.841,\ 5.331,\ 8.536,\ \ldots \\ m=2,\ \ x'_{2l} & = & 3.054,\ 6.706,\ 9.970,\ \ldots \\ m=3,\ \ x'_{3l} & = & 4.201,\ 8.015,\ 11.336,\ \ldots \end{array}$$

In this case the resonant frequencies are given by

$$\omega_{mln} = \frac{1}{\sqrt{\mu\epsilon}}\Big(\frac{x'^2_{ml}}{R^2} + \frac{n^2\pi^2}{d^2}\Big)^{1/2}$$

The lowest TE mode is TE_{111} and its resonance frequency is

$$\omega_{111} = \frac{1.841}{\sqrt{\mu\epsilon}}\frac{c}{R}\Big(1 + 2.912\frac{R^2}{d^2}\Big)^{1/2}$$

while the fields are derived from

$$\psi = H_z = H_0 J_1\Big(\frac{1.841\rho}{R}\Big)\cos\phi\sin\frac{\pi z}{d}e^{-i\omega t}$$

by substituting in (34). For d large enough ($d > 2.03R$), the resonance frequency ω_{111} is smaller than that for the lowest TM mode. Then the TE_{111} mode is the fundamental oscillation of the cavity. Because the frequency depends on the ratio d/R, it is possible to provide easy tuning by making the separation of the end faces adjustable. Finally, we must fix the accelerating electric field that we apply in the resonant cavities, so that we won't have deceleration. We suppose that

$$E = E_0 \sin \omega_E t$$

If we take the radius of the cyclic accelerator to be

$$r \simeq 17\ m$$

then the bundle will have a period

$$T_a = \frac{2\pi r}{v} \simeq 0.357\ \mu s$$

and frequency

$$\omega_a \simeq 17.588\ MHz.$$

So from this qualitative calculation the electric frequency must be

$$\omega_E = 2k\pi\omega_a, \quad k = 0, \pm 1, \pm 2, \ldots$$

The above result is a theoretical result. In reality we cannot achieve this values for the electric frequency. The result is that we have a stability problem in the simultaneity of the frequencies.

4.2 Earth and Ionosphere : A Big Resonant Cavity

In first order approximation, the Earth-atmosphere system can be seen from an electromagnetic point of view as a radial shell of three layers of conductivity. The Earth and the Ionosphere, in about 100km height, appear as a perfect conductor with the air of negligible conductivity in between. They form a spherical shell of conductivity, denoted Earth-Ionosphere cavity, in which EM radiation is trapped.

Lightning bolts containing a wide spectrum of frequencies act as sources of radial electric fields and lightning strikes within the troposphere, radiate energy into this system. So the Earth as one boundary surface and the ionosphere as the other, perform a spherical resonant cavity.

If we consider them as perfectly conducting concentric spheres of radii

$$a = 6400\ km$$

for Earth and

$$b = a + h = 6500\ km$$

for Ionosphere, since the second one, as we mentioned before, is approximately at a

$$h = 100\ km$$

height above the Earth, all we have to do is solve the math problem.

Furthermore, if we are concerned about only the lowest frequencies, we can focus our attention on the TM modes, with only tangential magnetic fields. The reason for this is that the lowest frequencies of TE modes are of the order of

$$\omega_{TE} \sim \frac{\pi c}{h} \quad ,$$

a lot higher than the lower frequencies of TM modes[15]

$$\omega_{TM} \sim \frac{c}{a} \quad .$$

This higher price can be explained because of the fact that the TE modes, with only tangential electric fields, must have a radial variation of half a wavelength between $r = a$ and $r = b$.

So combining the two Maxwell equations we get

$$\frac{\omega^2}{c^2}\vec{B} - \vec{\nabla} \times \vec{\nabla} \times \vec{B} = 0$$

The ϕ component of the above is

$$\frac{\omega^2}{c^2}(rB_\phi) + \frac{\partial^2}{\partial r^2}(rB_\phi) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta r B_\phi)\right] = 0$$

The last part can be transformed into

$$\frac{\partial}{\partial \theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta r B_\phi)\right] = \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}(rB_\phi)\right) - \frac{rB_\phi}{\sin^2\theta}$$

We see that the θ dependence is given by the associated Legendre polynomials

$$P_l^m(\cos\theta), \quad with \quad m = \pm 1$$

A solution will be

$$B_\phi(r, \theta) = \frac{u_l(r)}{r} P_l^1(\cos\theta)$$

[15] To understand this, one has first to solve the problem and find the modes with the resonant frequencies.

Substituting the above into the differential equation we now have a differential equation for u_l

$$\frac{d^2u_l(r)}{dr^2} + \left[\frac{\omega^2}{c^2} - \frac{l(l+1)}{r^2}\right] u_l(r) = 0$$

The radial and tangential electric fields are

$$E_r = \frac{ic}{\omega r \sin\theta} \frac{\partial}{\partial\theta}(\sin\theta B_\phi) = -\frac{ic}{\omega r} l(l+1) \frac{u_l(r)}{r} P_l^1(\cos\theta)$$

$$E_\theta = -\frac{ic}{\omega r} \frac{\partial}{\partial r}(r B_\phi) = -\frac{ic}{\omega r} \frac{\partial u_l(r)}{\partial r} P_l^1(\cos\theta)$$

As $E_\theta = 0$ at $r = a$, $r = b$ we see that the boundary condition for the differential equation of $u_l(r)$ is

$$\left.\frac{du_l(r)}{dr}\right|_{r=a,b} = 0$$

With the last boundary condition, a solution for $u_l(r)$ is

$$u_l(r) = A\cos[q(r-a)]$$

where

$$qh = n\pi \ .$$

Only for $n = 0$ we get the very low frequencies. If $n = 0$, then $q = 0$ and

$$u_l = constant \ .$$

So the characteristic resonant frequencies are

$$\omega_l = \sqrt{l(l+1)} \frac{c}{a} \tag{35}$$

These resonant frequencies are the so called *Schumann resonances*. The first seven values are the following 8, 14, 20, 26, 32, 37 and 43. Now one will ask:

> "What all this has to do with...understanding sleep patterns and relaxation levels of human beings??"

The natural frequencies of the Human Brain are:

- Beta waves (14-30 Hz)
- Alpha waves (8-13 Hz)
- Theta waves (4-7 Hz)
- Delta waves (1-3 Hz)

Alpha frequencies have been associated with meditation and relaxation. Theta frequencies have been associated with dreamy and creative states. According to a web page related to an October 2002 Physics News Update:

> "... EEGs can now record brainwaves without the need for electrodes to be inserted into the brain or even for them to be placed on the scalp. The figure shows a brainwave trace for periods when the eyes are open and when they are closed. The red regions respond to the alpha wave (eyes closed) at a frequency of around 9 Hz. (Courtesy University of Sussex) ...".

The Beta and Alpha waves (8-30 Hz) seem to correspond to the *Schumann resonances*: 8, 14, 20, 26, 32, 37 and 43 Hertz.

The 30 Hz high end of the Beta waves is roughly coincident with the frequency of cats' purrs: According to an 18 March 2001 article in the London Telegraph by David Harrison:

> "... the purring of cats is a "natural healing mechanism" ... between 27 and 44 hertz ... was the dominant frequency for a house cat, and 20-50 Hz for the puma, ocelot, serval, cheetah and caracal. This reinforces studies confirming that exposure to frequencies of 20-50 Hz strengthens human bones and helps them to grow. ... Almost all cats purr, including lions and cheetahs, though not tigers. ...".

Some experiments show connections between the brain states and resonant electromagnetic waves, raising the possibility that the Human Brain has evolved to be "in tune" with Planet Earth. Dolphin and Human Brains may contain BioMagnetite that could give them an electromagnetic sense that could provide a link between Brains and many types of electromagnetic phenomena, including but not limited to Schumann Resonance Phenomena.

Related Mind and Consciousness phenomena include Parapsychological Phenomena such as found by the Princeton Engineering Anomalies Research, the Boundary Institute, the Cognitive Sciences Laboratory (STARGATE), and such as described by Jack Sarfatti, Brian Josephson, and Jessica Utts.

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